NEL

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Chapter

7

Logarithmic Functions

LEARNING GOALS

You will be able to develop your algebraic and graphical reasoning skills by

- Determining the characteristics of logarithmic functions
- Estimating and determining the values of logarithmic expressions
- Understanding and applying the laws of logarithms
- Using logarithms to solve exponential equations
- Modelling situations using logarithmic functions and then interpreting the models

? Canadian musician and mathematician Daniel Snaith is playing with his band Caribou. The loudness of the music can be measured in decibels (dB). A measure of 11 dB is 10 times louder than a measure of 1 dB. At a typical rock concert, the loudness measured in the front row may be 115 dB. In comparison, the loudness measured in the back row may be 102 dB. How much louder is music heard in the front row of a typical rock concert, compared with the back row?

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7 Getting Started

Fibre-Optic Cables

Fibre-optic technology involves transferring data in the form of light through special cables. This technology is the preferred method for transmitting large amounts of data over large distances. Since the 1980s, underwater fibre-optic cables have been used to link all the continents, except Antarctica. A fibre-optic cable is capable of transferring large amounts of data simultaneously throughout its length, and the data signals at the receiver's end are of extremely high quality. In 2006, 99% of all international telecommunications traffic involved underwater fibre-optic cables, while only 1% involved satellites.

Despite the efficiency of data transfer using fibre optics, the magnitude of the original signal still deteriorates over the length of a cable due to scattering and absorption within the walls of the cable. Depending on the frequency of the light and the type of cable, the deterioration may be 37% per kilometre. This means that the magnitude of an input signal is reduced

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from 100% to 63% after 1 km. To help maintain the magnitude of the input signal, signal amplifiers are placed at regular intervals over the length of the cable. The signal usually needs to be amplified when the magnitude reaches 9×10^{-14}% of the input signal.

? What would be the typical spacing between signal amplifiers along the length of a fibre-optic cable?

A. Explain why the exponential function $P(d) = 100(0.63)^d$ gives the percent of the magnitude of the input signal, $P(d)$, over d km.

B. Graph $P(d)$ as a function of d. Describe its characteristics, including
- the number of d-intercepts,
- the P-intercept,
- the end behaviour,
- the domain,
- the range, and
- whether the function is increasing or decreasing.

C. What is the domain and range of the function, based on this context?

D. If the magnitude of a data signal was 9×10^{-14}% of the input signal, what was the length of the cable through which the signal travelled?

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. It is difficult to analyze data when the magnitudes of the numbers are very different.
2. The domain of an exponential function of the form $x = 10^y$ has restrictions, unlike the domain of a function of the form $y = 10^x$.
3. Exponential functions are the only type of functions that can be used to model rapid increases or rapid decreases in data values.

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7.1 Characteristics of Logarithmic Functions with Base 10 and Base e

YOU WILL NEED

- graphing technology

EXPLORE...

- Use benchmarks to estimate the solution to this equation:

$$120 = 10^y$$

logarithmic function

A function of the form

$$y = a \log_b x$$

where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.

GOAL

Investigate the characteristics of logarithmic functions with base 10 and base e from graphs and equations.

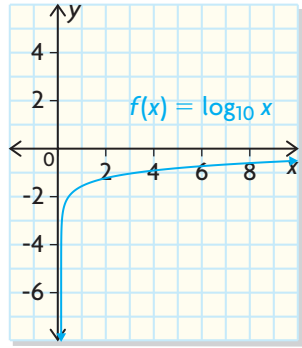
INVESTIGATE the Math

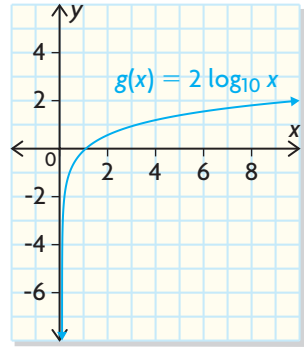
Randy came across a new type of function in his chemistry class. He learned that the pH of a solution is determined by a **logarithmic function**. He wondered how the characteristics of these functions differ from the characteristics of polynomial and exponential functions. Randy noticed that he had a log button on his calculator, so he looked in his user manual and discovered that this was a base 10 logarithm. Using this button, he created a table of values and a graph for each of the functions shown below to investigate the similarities and differences.

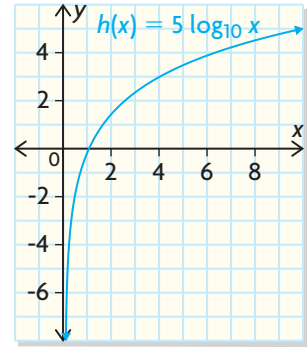
x	$f(x) = \log_{10} x$
−1	undefined
0	undefined
1	0
2	0.301...
3	0.477...
4	0.602...
5	0.698...
6	0.778...
7	0.845...
8	0.903...
9	0.954...
10	1

x	$g(x) = 2 \log_{10} x$
−1	undefined
0	undefined
1	0
2	0.602...
3	0.954...
4	1.204...
5	1.398...
6	1.556...
7	1.690...
8	1.806...
9	1.908...
10	2

x	$h(x) = 5 \log_{10} x$
−1	undefined
0	undefined
1	0
2	1.505...
3	2.385...
4	3.010...
5	3.494...
6	3.890...
7	4.225...
8	4.515...
9	4.771...
10	5







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? What are the characteristics of logarithmic functions of the form $y = a \log_b x$, where $b = 10$ or $b = e$, and a is a real number?

A. Use a graphing calculator to graph the logarithmic function $y = \log_{10} x$. On the same axes, graph $y = 10^x$. How are these two graphs related?

B. Randy choose three functions of the form $y = a \log_{10} x$, where $a > 0$. On a new screen, graph Randy's functions, shown on the previous page.

C. Examine the graph of each function, and state the following characteristics:
- the number of x-intercepts
- the y-intercept
- the end behaviour
- the domain
- the range

D. Examine the graphs you created in part B. As you move from left to right along the x-axis, do the y-values of each function increase or decrease? Then examine the table of values for each function on your graphing calculator. What happens to the y-values as the x-values increase?

E. Choose three new functions of the form $y = a \log_{10} x$, where $a < 0$. Repeat parts C and D for these new functions.

F. On a new screen, graph the natural logarithmic function $y = \ln x$. On the same axes, graph $y = e^x$. How are these two graphs related?

G. On a new screen, graph the function $y = \ln x$ and two other functions of the form $y = a \ln x$, where $a > 0$. Examine the graph of each function, and state the following characteristics:
- the number of x-intercepts
- the y-intercept
- the end behaviour
- the domain
- the range

H. Examine the graphs you created in part G. As you move from left to right along the x-axis, do the y-values of each function increase or decrease? Then examine the table of values for each function on your graphing calculator. What happens to the y-values as the x-values increase?

I. Choose three new functions of the form $y = a \ln x$, where $a < 0$. Repeat parts G and H for these new functions.

Communication *Tip*

The expression $\log_{10} x$ is known as the common logarithm or a logarithm with a base of 10. The expression is often written without the 10, so the two functions $y = \log_{10} x$ and $y = \log x$ are equivalent.

Communication *Tip*

The function $y = \log_{10} x$ is equivalent to $x = 10^y$, so a logarithm is an exponent. The meaning of $\log_{10} x$ is "the exponent that must be applied to base 10 to get the value of x." For example, $\log_{10} 100 = 2$.

Communication *Tip*

A logarithm with base e is called the natural logarithm and is written as $\ln x$. The functions $y = \log_e x$, $y = \ln x$, and $x = e^y$ are equivalent.

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Reflecting

J. Based on your observations, which characteristics of the graphs of logarithmic functions are similar to the characteristics of graphs of exponential functions you have studied?

K. Based on your observations, which characteristics of the graphs of logarithmic functions differ from the characteristics of graphs of exponential functions you have studied?

L. Can you predict the end behaviour of functions of the form $y = a \log x$ or $y = a \ln x$ based on the parameter a? Explain.

M. In logarithmic functions of the form $y = a \log_b x$, did changing the base from $b = 10$ to $b = e$ cause any of the characteristics to change? Explain.

APPLY the Math

EXAMPLE 1 Connecting the characteristics of an increasing logarithmic function to its equation and graph

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the following function:

$$y = 15 \log x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

Sid's Solution

The function $y = 15 \log x$ is a logarithmic function of the form $y = a \log_b x$, with $a = 15$ and $b = 10$.

I knew that this is a common logarithmic function since the base is not written. Its base is 10.

I determined the values of the parameters a and b.

Common Characteristics

x-intercept: 1
Number of y-intercepts: 0
Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
Range: $\{y \mid y \in \mathrm{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for all logarithmic functions of the form

$y = a \log_b x$, where $b = 10$.

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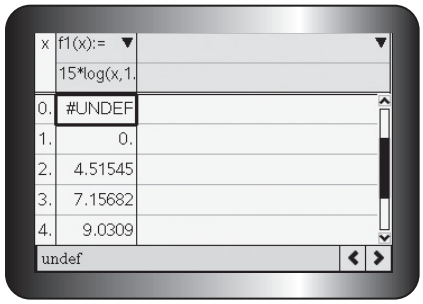
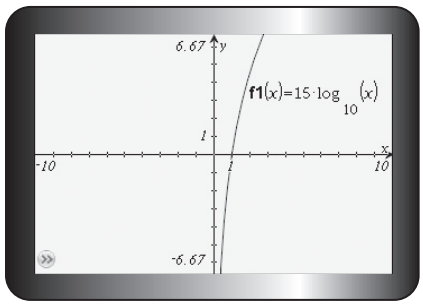
Unique Characteristics

End behaviour: The curve extends from quadrant IV to quadrant 1.

Since the parameter *a* is greater than 0, I knew that the function increases as you move from left to right along the *x*-axis.



I used a graphing calculator to create a table of values and a graph to verify my predictions.



My predictions were correct.

Your Turn

Predict the *x*-intercept, the number of *y*-intercepts, the end behaviour, the domain, and the range of this function:

$$y = -5 \log x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

EXAMPLE 2 Connecting the characteristics of a decreasing natural logarithmic function to its equation and graph

Predict the *x*-intercept, the number of *y*-intercepts, the end behaviour, the domain, and the range of the following function:

$$y = -4 \ln x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

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Anne's Solution

The function $y = -4 \ln x$ is a logarithmic function of the form $y = a \log_b x$ with a base of e. In this function, $a = -4$ and $b = e$.

I knew that $y = \ln x$ is the natural logarithmic function, which is equivalent to $y = \log_e x$, so I knew the base of the logarithmic function is 2.718…. I determined the values of the parameters a and b.

Common Characteristics

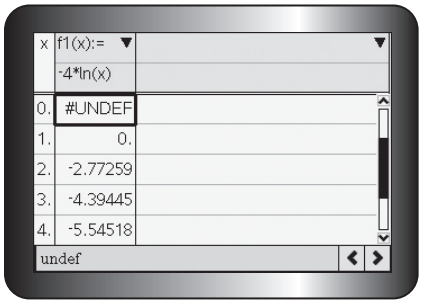
x-intercept: 1
Number of y-intercepts: 0
Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
Range: $\{y \mid y \in \mathrm{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for all logarithmic functions of the form $y = a \log_b x$, where $b = e$.

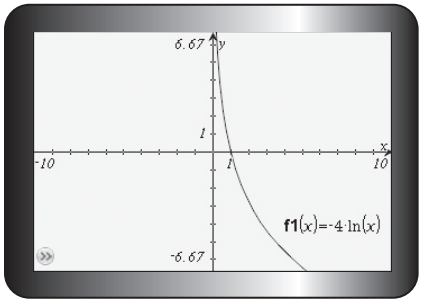
Unique Characteristics

End behaviour: The curve extends from quadrant I to quadrant IV.

Since the parameter a is less than 0, I knew that the function decreases as you move from left to right along the x-axis.



I used a graphing calculator to create a table of values and a graph.



My predictions were correct.

Your Turn

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of this function:

$$y = 12 \ln x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

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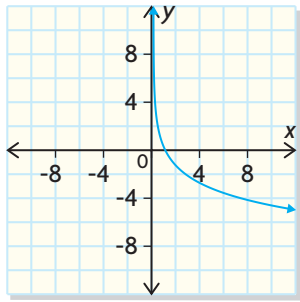
EXAMPLE 3 Matching equations of exponential and logarithmic functions with their graphs

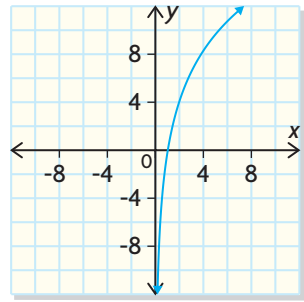
Which function matches each graph below? Provide your reasoning.

i) $y = 5(2)^x$ **ii)** $y = 2(0.1)^x$ **iii)** $y = 6 \log x$ **iv)** $y = -2 \ln x$

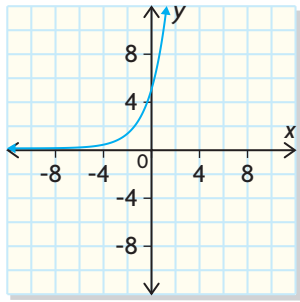
a)



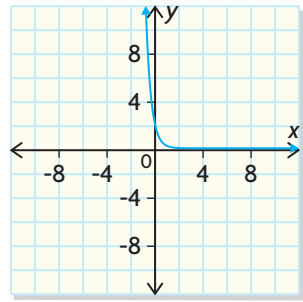
c)



b)



d)

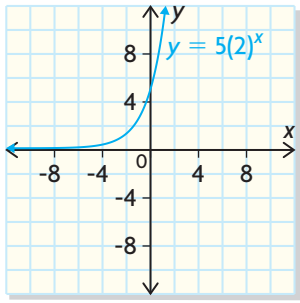


Erryn's Solution

Exponential functions

I know that graphs b) and d) are exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, since both extend from quadrant II to quadrant I.

I also know that functions i) and ii) are exponential, since the exponent in each equation is a variable.



Since $b > 0$ in the function $y = 5(2)^x$, I knew that this function matches the graph of the increasing function.

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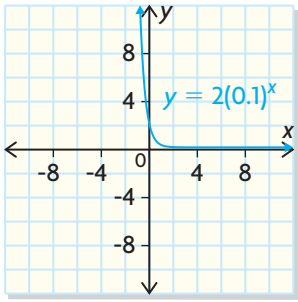


Since $b < 0$ in the function $y = 2(0.1)^x$, I knew that this function matches the graph of the decreasing function.

$y = 5(2)^x$
y-intercept: 5

I knew that I was right since the y-intercepts for the equations match the y-intercepts on the graphs.

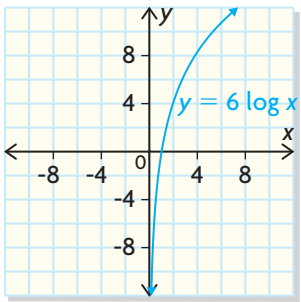
$y = 2(0.1)^x$
y-intercept: 2

Function i) matches graph b) and function ii) matches graph d).

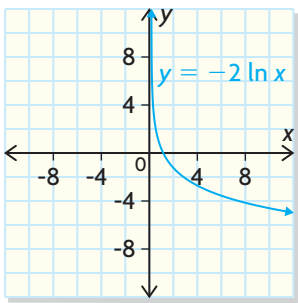
Logarithmic functions

I know that graphs a) and c) are logarithmic, since they extend from quadrant I to quadrant IV and from quadrant IV to quadrant I, respectively.

I also know that functions iii) and iv) are logarithmic.



Since $a > 0$ in the function $y = 6 \log x$, I knew that this function matches the graph of the increasing function.



Since $a < 0$ in the function $y = -2 \ln x$, I knew that this function matches the graph of the decreasing function.

Function iii) matches graph c) and function iv) matches graph a).

Your Turn

Jordan claims that she can tell which graphs are exponential and which graphs are logarithmic by examining the domain of each function. Do you agree or disagree? Explain.

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In Summary

Key Ideas

- A logarithmic function has the form $f(x) = a \log_b x$, where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.
- All logarithmic functions of the form $f(x) = a \log x$ and $f(x) = a \ln x$ have these characteristics:

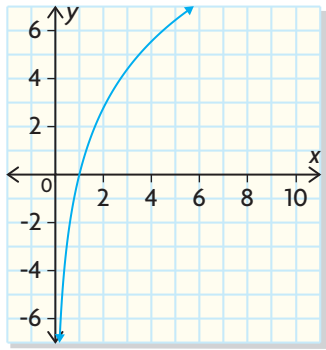
x-Intercept	1
Number of y-Intercepts	0
End Behaviour	The curve extends from quadrant IV to quadrant I or quadrant I to quadrant IV.
Domain	$\{x \mid x > 0, x \in \mathrm{R}\}$
Range	$\{y \mid y \in \mathrm{R}\}$

- All logarithmic functions of the form $f(x) = a \log x$ and $f(x) = a \ln x$ have these unique characteristics:
  - If $a > 0$, the function increases.
  - If $a < 0$, the function decreases.

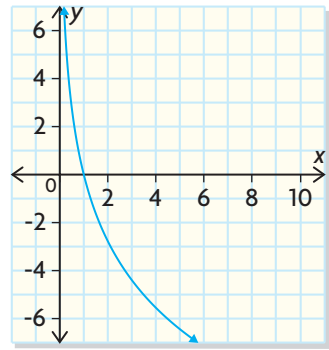
Need to Know

- The graph of a logarithmic function of the form $f(x) = a \log x$ or $f(x) = a \ln x$ will look like one of the following cases:

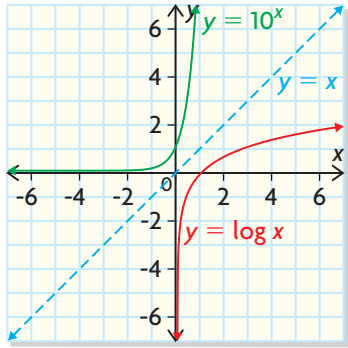
Case 1: an increasing function, where $a > 0$



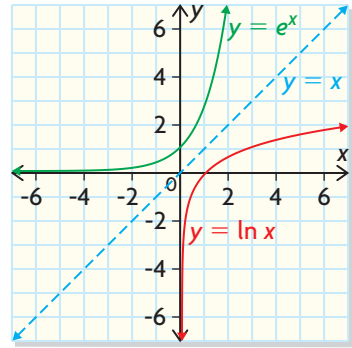
Case 2: a decreasing function, where $a < 0$



- The graph of $y = \log x$ is a reflection of the graph of $y = 10^x$ about the line $y = x$.



- The graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ about the line $y = x$.



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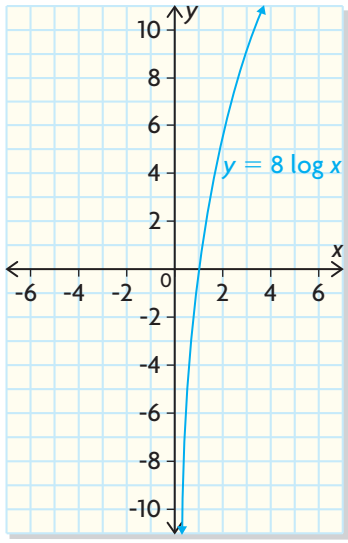
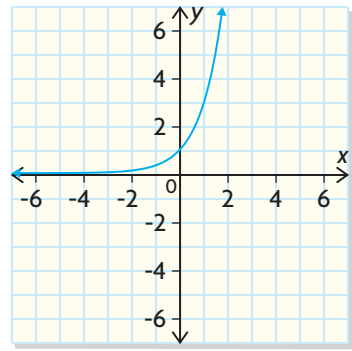
CHECK Your Understanding



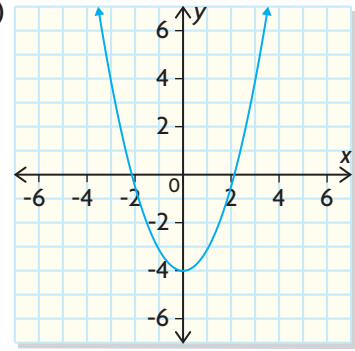
1. State the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the function shown on the left.

2. Identify which of these graphs are logarithmic functions. Explain how you know.

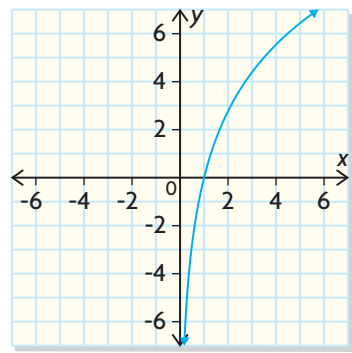
a)



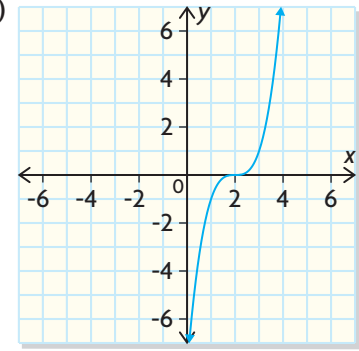
b)



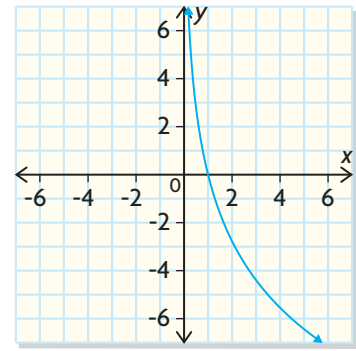
c)



d)



e)



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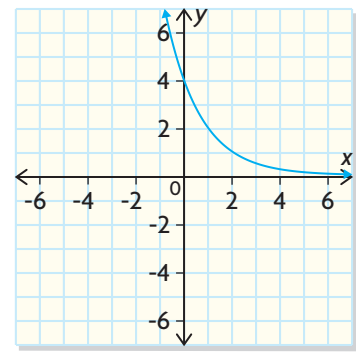

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3. For the functions in question 2 that are logarithmic, state the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range. Also state whether the parameter a in the equation of the function is greater than zero or less than zero, and explain how you know.

PRACTISING

4. Use graphing technology to graph the functions $y = \log x$ and $y = -\log x$. Compare the graphs of the functions. Consider the x-intercept, the number of y-intercepts, the end behaviour, the domain, the range, and whether they are increasing or decreasing.

5. Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, the range, and whether each logarithmic function is increasing or decreasing, based on its equation. Verify your predictions using graphing technology.

a) $f(x) = 3 \log x$

b) $f(x) = -9 \log x$

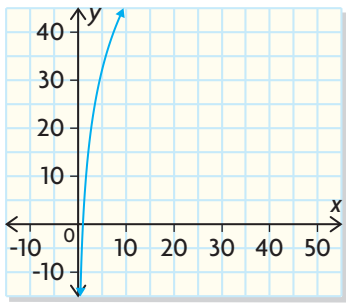
c) $f(x) = 12 \ln x$

d) $f(x) = 0.5 \ln x$

e) $f(x) = -0.2 \log x$

f) $f(x) = 100 \ln x$

6. State three reasons why you know that the following graph represents a logarithmic function of the form $y = a \log x$.



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7. Sketch the graph of the logarithmic function with each set of characteristics.

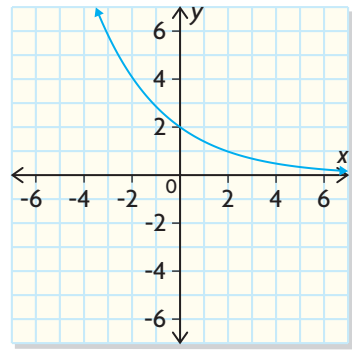
a) x-intercept: 1
Number of y-intercepts: 0
End behaviour: The curve extends from quadrant IV to quadrant I.
Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
Range: $\{y \mid y \in \mathrm{R}\}$

b) x-intercept: 1
Number of y-intercepts: 0
End behaviour: The curve extends from quadrant I to quadrant IV.
Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
Range: $\{y \mid y \in \mathrm{R}\}$

8. Match each function with its corresponding graph. Provide your reasoning.

i) $y = 4.2 \log x$
ii) $y = -3 \log x$
iii) $y = 6^x$
iv) $y = 2(0.7)^x$

a)



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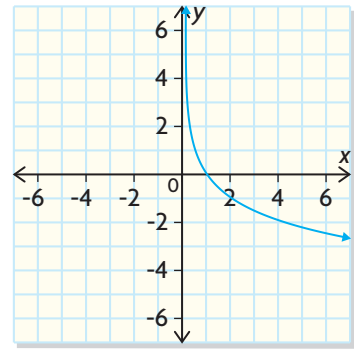

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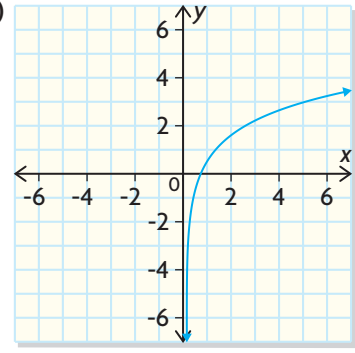

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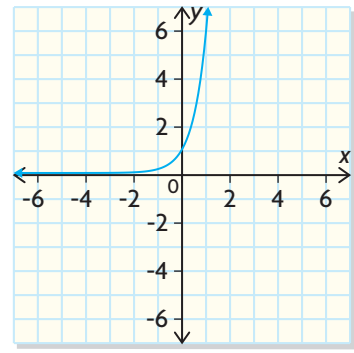

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9. Holly claims that she can distinguish between the graph of an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, and the graph of a logarithmic function of the form $y = a \log x$, where $a \neq 0$, based entirely on the number of x-intercepts shown. Do you agree or disagree? Explain.

10. The pH of a solution, $p(x)$, can be determined using the function

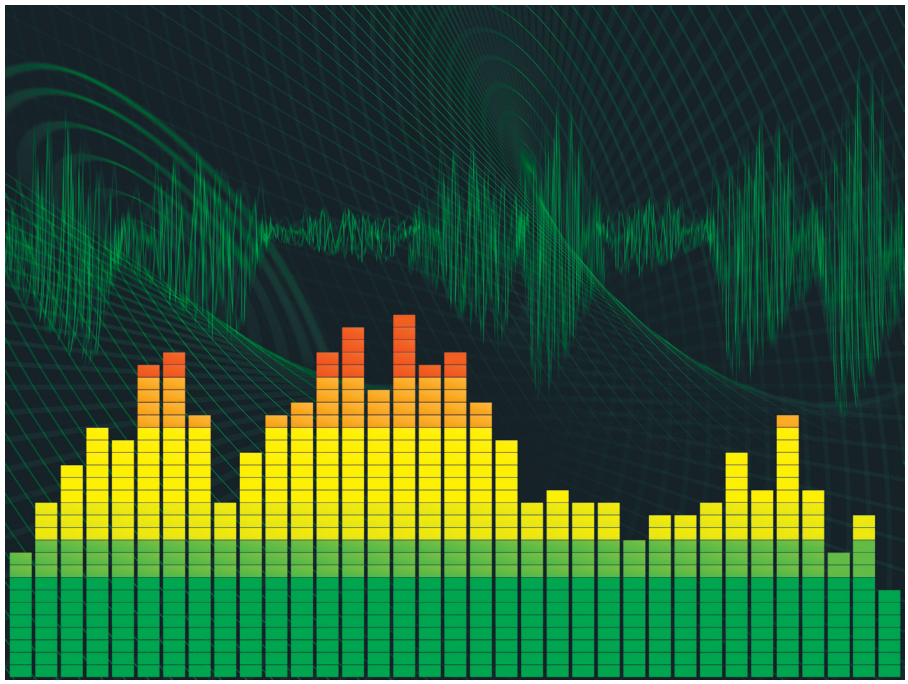
$$p(x) = -\log x$$

where x represents the concentration of hydrogen ions in the solution, in moles per litre. Use the equation of the function to predict what happens to the pH of a solution as the concentration of hydrogen ions increases. Verify your prediction using graphing technology.

11. Sound intensity is measured on the decibel scale. A decibel is $\frac{1}{10}$ of a bel, named after Alexander Graham Bell. The intensity of a sound (number of decibels) can be determined using the function

$$D(x) = 10 \log (10^{16} \cdot x)$$

where x represents the energy of the sound in watts per square centimetre (watts/cm^2). Use the equation of the function to predict what happens to the number of decibels, $D(x)$, as the energy of the sound, x, increases. Verify your prediction using graphing technology.



A recording of speech. Like other sounds, speech can be measured by pitch (top graph) and by intensity (bottom graph).

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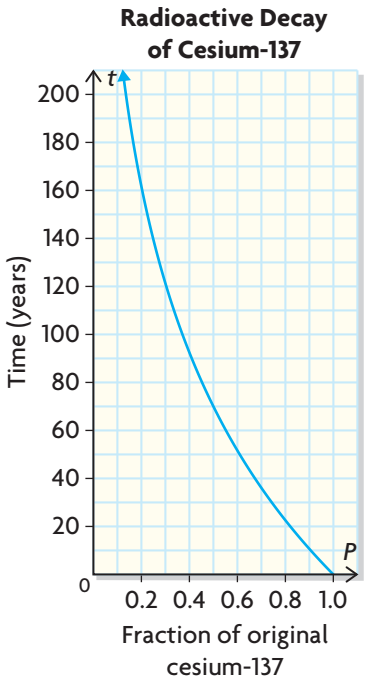

12. Cesium-137 is created in nuclear reactors. It decays naturally over time. The approximate time for cesium-137 to decay is given by the function

$$t = -100.422 \log P$$

where t represents the time in years and P represents the percent of the original amount of cesium-137, as a decimal, that remains. The function is graphed at the left.

a) State the following characteristics of the graph:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing
- the value of the function at $P = 1$

b) A sample of cesium-137 is taken from a reactor. Estimate the time when 5% of the sample will remain.

c) Estimate the time when 50% of the sample will remain.

Closing

13. Consider the following functions:

$$y = a \log x \quad \text{and} \quad y = a \ln x$$

a) For what values of a is each function decreasing?

b) Do these functions have an unrestricted domain? Explain.

c) Do these functions have an unrestricted range? Explain.

Extending

14. The largest known earthquake in Saskatchewan occurred in 1909. Based on recorded observations, the earthquake is believed to have measured approximately 5.5 on the Richter scale. The magnitude, M, of an earthquake that is T times more intense than an earthquake registering 5.5 on the Richter scale is given by the following function:

$$M = \log T + 5.5$$

For every 1 unit increase on the Richter scale, the intensity of an earthquake increases by a factor of 10.

a) Use technology to graph the function, and state the following characteristics of the graph:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing
- the value of the function at $T = 1$

b) Determine the magnitude of an earthquake that is 50 times more intense than the Saskatchewan earthquake.

c) In 2011, an earthquake that measured 9.0 on the Richter scale occurred off the coast of Japan, creating a devastating tsunami. About how many times more intense was this earthquake than the Saskatchewan earthquake? Use your graph to estimate.

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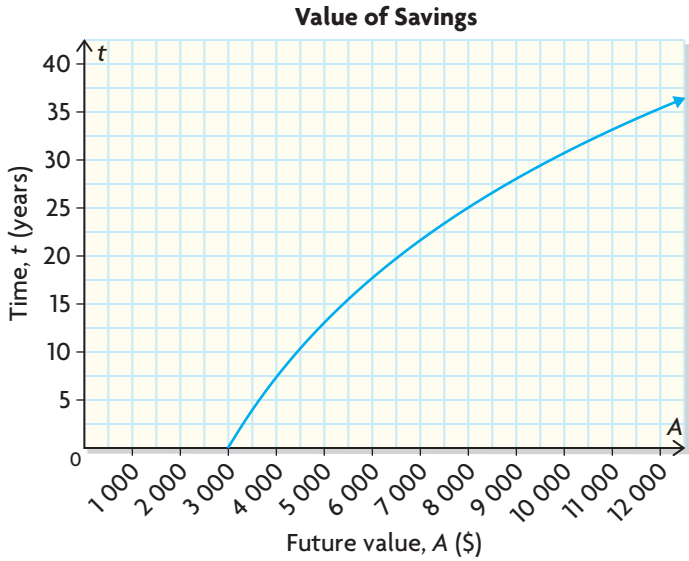
15. Christie has $3000 invested in a savings account that earns 4%/a. The approximate time for her investment to reach a given future value is modelled by the function

$$t = 58.7 \log A - 204.1$$

where t is the time in years and A is the future value. The function is graphed below.



a) State the following characteristics of this function:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing

b) How long, in years, will it take for Christie's investment to reach $10 000?

c) How long, in years, will it take for Christie's investment to double?

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7.2 Evaluating Logarithmic Expressions

YOU WILL NEED

- graphing technology

EXPLORE...

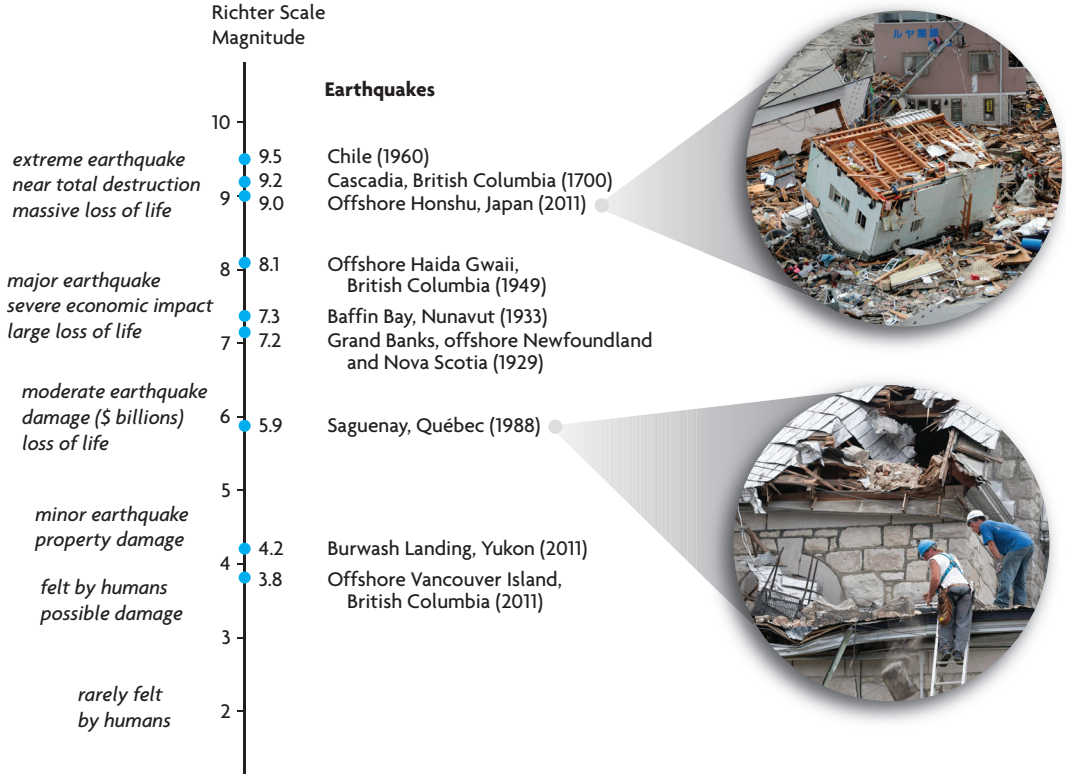
- Consider powers of the form b^x, where $b > 1$, $x > 1$, and both b and x are whole numbers. What combinations of values of b and x will result in a value of b^x that is less than 100?

GOAL

Develop strategies to evaluate logarithmic expressions, and interpret problems that involve logarithms.

LEARN ABOUT the Math

David learned that the Richter scale is used to measure the amount of energy released, or magnitude, of an earthquake. According to the Geological Survey of Canada, more than 4000 earthquakes occur in Canada every year, but only about 50 of these earthquakes are felt by humans.



The magnitude of an earthquake, y, can be calculated using the logarithmic function

$$y = \log x$$

where x is the amplitude of the vibrations, measured by a seismograph.

? How can the magnitude of an earthquake be determined using the function $y = \log x$, and how can this function be written in an exponential form?

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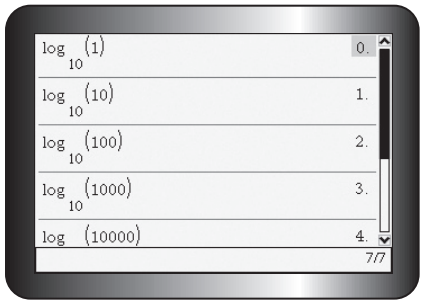
EXAMPLE 1 Connecting logarithms to exponents

Use the function $y = \log x$ to complete the table of values below.
Then express the function in an exponential form.

Amplitude of Vibrations Measured by Seismograph, *x* (units)	1	10	100	1000	10 000	100 000	1 000 000
Richter Scale Magnitude, *y*							

Debbie's Solution

I used the log button on my calculator to evaluate the function for each given value of x.



The base of the logarithm is not included, so the function is a common logarithm with a base of 10.

These are the Richter scale magnitudes for the seismograph readings in the table.

I completed the table of values and noticed a pattern.

	10^0	10^1	10^2	10^3	10^4	10^5	10^6
Amplitude of Vibrations Measured by Seismograph, *x* (units)	1	10	100	1000	10 000	100 000	1 000 000
Richter Scale Magnitude, *y*	0	1	2	3	4	5	6

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If each x-value is written as a power of 10, there is a relationship between the y-value and the exponent on each power of 10.

$\log 1 = 0$ because $10^0 = 1$
$\log 10 = 1$ because $10^1 = 10$
$\log 100 = 2$ because $10^2 = 100$
$\log 1000 = 3$ because $10^3 = 1000$
$\log 10\,000 = 4$ because $10^4 = 10\,000$
$\log 100\,000 = 5$ because $10^5 = 100\,000$
$\log 1\,000\,000 = 6$ because $10^6 = 1\,000\,000$

Therefore, to evaluate the logarithmic expression $\log_{10} x$, I reasoned that I had to determine the exponent that must be applied to base 10 to get each value of x. I tested my conjecture for each value of x in the table.

My conjecture seems reasonable.

Using inductive reasoning to generalize, if I am given a function in logarithmic form,

$y = \log_{10} x$,

I can write the function in this equivalent exponential form:

$x = 10^y$

I tested my conjecture using my calculator for other seismograph readings.

When $x = 500$,
$\log 500 = 2.698...$ and
$10^{2.698...} = 500$

When $x = 2500$,
$\log 2500 = 3.397...$ and
$10^{3.397...} = 2500$

I can conclude that the function $y = \log_{10} x$ is equivalent to $x = 10^y$.

Reflecting

A. In the previous lesson, you investigated the characteristics of the natural logarithmic function, $y = \ln x$.
- **i)** What is the base of this function?
- **ii)** Make a conjecture about how this function could be written in its equivalent exponential form.
- **iii)** Use your calculator to test your conjecture when $x = 5$ and when $x = 100$.

B. The general logarithmic function is defined as $y = \log_b x$, where $b > 0$ and $b \neq 1$. How could you write this function in its equivalent exponential form?

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APPLY the Math

EXAMPLE 2 | Evaluating and interpreting an exponent

Determine the value of y in each exponential equation. Verify your answer.

a) $81 = 10^y$ **b)** $25 = e^y$

Matthew's Solution

a) $81 = 10^y$

$\log_{10} 81 = y$

I knew that an exponential function in the form $x = b^y$ is equivalent to a logarithmic function in the form $y = \log_b x$. I could not write 81 as a power of 10, so I wrote the exponential equation in logarithmic form.

The base of the logarithm form, 10, is the same as the base of the exponential form. Since a logarithm is an exponent, and y is the exponent in the given equation, y must be equal to $\log_{10} 81$.

$1.908... = y$

The base of the logarithm is 10, so I used my calculator to evaluate.

Verify $y = 1.908....$

LS:	RS:
81	$10^{1.908...}$
	81

To verify, I substituted my answer into the original equation.

$LS = RS$

b) $25 = e^y$

$\log_e 25 = y$

I knew that I could not write 25 as a power of e, so I wrote the given exponential equation in logarithmic form.

$\ln 25 = y$

$3.218... = y$

I knew that a logarithm with a base of e is equivalent to the natural logarithm, ln x. I could evaluate the value of ln 25 using my calculator.

Verify $y = 3.218....$

LS:	RS:
25	$e^{3.218...}$
	25

To verify, I substituted my answer into the original equation.

$LS = RS$

Your Turn

Determine the value of y in the following exponential equation:

$$32 = 10^y$$

Verify your answer.

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EXAMPLE 3 Evaluating and interpreting a logarithmic expression

Evaluate each logarithmic expression.

a) $\log_2 16$ **b)** $\log_4 64$

Trevor's Solution

a) $\log_2 16$

$\log_2 16 = y$ — I knew that a logarithm is an exponent. In the given logarithm, the base is 2. To evaluate the logarithm, I set it equal to y.

$2^y = 16$ — I rewrote the equation in exponential form. I needed to determine the exponent of 2 that would give a value of 16.

$2^y = 2^4$ — I rewrote the equation with the same base.

$y = 4$ — Since the bases are the same, the exponents must be equal.

$\log_2 16 = 4$

b) $\log_4 64$

$\log_4 64 = y$ — This logarithm has a base of 4. To evaluate the logarithm, I set the expression equal to y.

$4^y = 64$ — I rewrote the equation in exponential form. I needed to determine the exponent of 4 that would give a value of 64.

$4^y = 4^3$ — I rewrote the equation with the same base.

$y = 3$ — Since the bases are the same, the exponents must be equal.

$\log_4 64 = 3$

Your Turn

Evaluate $\log_3 27$.

EXAMPLE 4 Evaluating single logarithmic expressions by writing the expressions in exponential form

Evaluate each logarithmic expression without technology.

a) $\log_3 \left(\frac{1}{27}\right)$ **b)** $\log_{\left(\frac{1}{4}\right)} 64$ **c)** $\log_2 (-4)$

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Liam's Solution

a) $\log_3\left(\frac{1}{27}\right) = y$

$3^y = \frac{1}{27}$

I set the expression equal to y and then rewrote the equation in exponential form.
I needed to determine the exponent of 3 that gives a value of $\frac{1}{27}$.

$3^y = 3^{-3}$

I knew that $\frac{1}{27}$ is 3^{-3}. I rewrote the equation with the same base.

$y = -3$

Since the bases are the same, the exponents must be equal.

b) $\log_{(\frac{1}{4})} 64 = y$

To evaluate the expression, I set it equal to y.

$\left(\frac{1}{4}\right)^y = 64$

I rewrote the equation in exponential form. I noticed that I needed to determine the exponent of $\frac{1}{4}$ that gives a value of 64.

$(4^{-1})^y = 4^3$

I knew that $\frac{1}{4}$ can be written as 4^{-1}. I also knew that 4^3 is 64. I rewrote the equation with the same bases.

$4^{-y} = 4^3$

I simplified the left side by applying the power law for exponents.

$-y = 3$

Since the bases are the same, the exponents must be equal.

$y = -3$

c) $\log_2(-4)$

$\log_2(-4) = y$

$2^y = -4$

If I set this expression equal to y and rewrite the equation in exponential form, the resulting power is equal to a negative number. This is not possible.

A positive number raised to any exponent cannot result in a negative answer.

This logarithm is undefined.
A logarithmic function has domain $x > 0$, where $x \in \mathrm{R}$.

I knew that the domain of a logarithmic function in the form $y = \log_b x$ is restricted to numbers that are greater than zero. Therefore, to take the logarithm of a number, the number cannot be zero or negative.

Your Turn

Evaluate $\log_5\left(\frac{1}{25}\right)$ without technology.

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EXAMPLE 5 Evaluating logarithmic expressions that contain two logarithms

Which expression has the greater value?

A: $\log_2 16 + \log_2 2$ **B:** $\log_2 1 - \log_2 \left(\frac{1}{8}\right)$

Hannah's Solution

A: $\log_2 16 + \log_2 2$

I decided to evaluate expression A first and to evaluate each term separately.

First term: $\log_2 16$
$2^4 = 16$, so the first term in expression A is equal to 4.

$\log_2 16$ means "the exponent applied to base 2 that gives a value of 16."

Second term: $\log_2 2$
$2^1 = 2$, so the second term in expression A is equal to 1.

$\log_2 2$ means "the exponent applied to base 2 that gives a value of 2."

$\log_2 16 + \log_2 2 = 4 + 1$
$= 5$

I substituted my results into the expression and evaluated the expression.

B: $\log_2 1 - \log_2 \left(\frac{1}{8}\right)$

I evaluated expression B next, by evaluating each term separately.

First term: $\log_2 1$
$2^0 = 1$, so the first term in expression B is equal to 0.

$\log_2 1$ means "the exponent applied to base 2 that gives a value of 1."

Second term: $\log_2 \left(\frac{1}{8}\right)$

$2^{-3} = \frac{1}{8}$, so the second term in expression B is equal to -3.

$\log_2 \left(\frac{1}{8}\right)$ means "the exponent applied to base 2 that gives a value of $\frac{1}{8}$."

$\log_2 1 - \log_2 \left(\frac{1}{8}\right) = 0 - (-3)$
$= 3$

I substituted my results into the expression and evaluated the expression.

$5 > 3$
$\log_2 16 + \log_2 2$ has the greater value.

I compared my results.

Your Turn

Is expression C greater or less than expression A in Example 5? Justify your decision.

C: $\log_2 64 \div \log_2 8$

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EXAMPLE 6 Using logarithms to determine and compare pH levels

The pH scale in chemistry is used to measure the acidity of a solution. The pH scale is logarithmic, with a base of 10. A logarithmic scale is useful for comparing numbers that vary greatly in size. The pH, $p(x)$, is defined by the equation

$$p(x) = -\log x$$

where the concentration of hydrogen ions, x, in a solution is measured in moles per litre (mol/L).

		Examples of solutions
acidity increases in this direction ↑	pH = 0	acid used in car batteries
	pH = 1	acid secreted by lining of stomach
	pH = 2	lemon juice, vinegar
	pH = 3	grapefruit, orange juice, pop
	pH = 4	tomato juice, acid rain
	pH = 5	soft drinking water, black coffee
	pH = 6	saliva
neutral	pH = 7	distilled water
	pH = 8	seawater
	pH = 9	baking soda
alkalinity increases in this direction ↓	pH = 10	milk of magnesia
	pH = 11	ammonia solution used in cleaning products
	pH = 12	soapy water
	pH = 13	bleaches, oven cleaner
	pH = 14	chemical used in liquid drain cleaner

A neutral solution, such as distilled water, has a pH of 7. An acidic solution, such as pop, has a pH that is less than 7. The lower the pH, the more acidic the solution is. An alkaline solution, such as milk of magnesia, has a pH that is greater than 7. The higher the pH, the more alkaline the solution is.

An increase or decrease of one unit on the pH scale indicates a change in the hydrogen ion concentration by a factor of 10. For example, a solution with a pH of 4 has 10 times the hydrogen ion concentration of a solution with a pH of 5 and, as a result, is 10 times more acidic.

a) The hydrogen ion concentration, x, of a solution is 0.0001 mol/L. Calculate the pH of the solution.

b) Use the pH scale shown to calculate the hydrogen ion concentration of lemon juice.

c) In terms of hydrogen ion concentration, how much more acidic is a solution with a pH of 1.6 than a solution with a pH of 2.5? Round your answer to the nearest tenth.

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Tara's Solution

a) $p(x) = -\log x$

$p(x) = -\log 0.0001$

$p(x) = 4$

The solution has a pH of 4.

I substituted the value of the hydrogen ion concentration, 0.0001 mol/L, into the pH equation.

Since the pH equation involves the common logarithm, I used my calculator to evaluate the expression on the right side.

b) $p(x) = -\log x$

$2 = -\log x$

$-2 = \log x$

$x = 10^{-2}$ or $\frac{1}{100}$

From the pH scale, I knew that lemon juice has a pH of 2. I substituted this value into the pH equation.

Since I needed to solve for x, I wrote the equation in exponential form.

The hydrogen ions in lemon juice is 0.01 mol/L.

c) Solution A:

$p(x) = -\log x_A$

$1.6 = -\log x_A$

$-1.6 = \log x_A$

$x_A = 10^{-1.6}$

Solution B:

$p(x) = -\log x_B$

$2.5 = -\log x_B$

$-2.5 = \log x_B$

$x_B = 10^{-2.5}$

I used the pH equation to calculate the hydrogen ion concentration in each solution.

I wrote both equations in exponential form.

$\frac{x_A}{x_B} = \frac{10^{-1.6}}{10^{-2.5}}$

To determine the relative acidity of solutions A and B, I set up a ratio.

$\frac{x_A}{x_B} = 10^{(-1.6-(-2.5))}$

I used the quotient law for exponents to simplify.

$\frac{x_A}{x_B} = 10^{(-1.6+2.5)}$

$\frac{x_A}{x_B} = 10^{0.9}$

$\frac{x_A}{x_B} = 7.943\ldots$

I used a calculator to evaluate the expression on the right. Since $10^{0.9}$ is less than 10, this value is reasonable.

Solution A is about 7.9 times more acidic than solution B.

Your Turn

Use the values on the pH scale to answer each question.

a) What is the hydrogen ion concentration of soapy water?

b) In terms of hydrogen ion concentration, how much more acidic is seawater than a soapy water solution?

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In Summary

Key Ideas

- The logarithmic function

$$y = \log_b x$$

is equivalent to the following exponential function:

$$x = b^y$$

- A logarithm is an exponent. The expression $\log_b x$ means "the exponent that must be applied to base b to give the value of x. For example:

$$\log_2 8 = 3$$

since

$$2^3 = 8$$

- The value of a logarithm can be determined in one of the following ways:
  - Set the logarithmic expression equal to y, and write the equivalent exponential form. Then determine the exponent to which the base must be raised to get the required number.
  - If the base of the logarithm is 10 or e, you can use a scientific or graphing calculator.

Need to Know

- The common logarithmic function

$$y = \log x$$

is equivalent to the following exponential function:

$$x = 10^y$$

- The natural logarithmic function

$$y = \ln x$$

is equivalent to the following exponential function:

$$x = e^y$$

- The logarithm of a negative number does not exist, because a negative number cannot be written as a power of a positive base.
- Many real-life situations have values that vary greatly. A logarithmic scale with powers of 10 can be used to make comparisons between large and small values more manageable. Three examples of logarithmic scales are the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution), and the decibel scale (used to measure sound level).

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CHECK Your Understanding

1. Write each logarithmic function in exponential form.

a) $y = \log x$ **b)** $y = \ln x$ **c)** $y = \log_5 x$ **d)** $y = \log_8 x$

2. Write each exponential equation in logarithmic form.

a) $x = e^y$ **b)** $x = 10^y$ **c)** $x = 4^y$ **d)** $x = 9^y$

3. Evaluate each logarithmic expression, to the nearest hundredth, with technology.

a) $\log 20$ **b)** $\log 45$ **c)** $\ln 40$ **d)** $\ln 98$

4. Evaluate each logarithmic expression without technology.

a) $\log_6 36$ **b)** $\log_4 64$ **c)** $\log_{10} 1000$ **d)** $\log_2 16$

PRACTISING

5. Estimate the value of y in each equation. Check your estimate by writing the equation in logarithmic form and evaluating with a calculator.

a) $250 = 10^y$ **b)** $9 = e^y$

6. Write each equation in logarithmic form.

a) $\frac{1}{9} = 3^y$ **c)** $5 = 7^y$

b) $1\,000\,000 = 2^y$ **d)** $x = a^y$

7. Write each logarithmic equation in exponential form.

a) $x = \log_{(\frac{1}{2})} 100$ **c)** $x = \ln 0.25$

b) $x = \log_{20} 40$ **d)** $x = \ln 1000$

8. Evaluate each logarithmic expression without technology.

a) $\log_3 81$ **b)** $\log_4 16$ **c)** $\log_8 64$ **d)** $\log_5 1$

9. Evaluate each logarithmic expression without technology.

a) $\log_3\left(\frac{1}{9}\right)$ **b)** $\log_2\left(\frac{1}{8}\right)$ **c)** $\log_4\left(\frac{1}{64}\right)$ **d)** $\log_5\left(\frac{1}{125}\right)$

10. Evaluate each logarithmic expression without technology.

a) $\log_{(\frac{1}{3})} 9$ **c)** $\log_{(\frac{1}{2})}\left(\frac{1}{4}\right)$

b) $\log_{(\frac{1}{2})} 8$ **d)** $\log_{(\frac{1}{7})}\left(\frac{1}{49}\right)$

11. Evaluate each expression without technology.

a) $\log_3 81 + \log_3 3$ **c)** $(\log_4 1)\left(\log_5\left(\frac{1}{5}\right)\right)$

b) $\log_2 64 - \log_2 4$ **d)** $\log_3 27 \div \log_3\left(\frac{1}{9}\right)$

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12. Which expression is greater?

A. $\log_2 32 - \log_2 8$ **B.** $\log 85 + \log 5$

13. Order these expressions from least to greatest. Provide your reasoning.

A. $\log\left(\frac{1}{2}\right)\left(\frac{1}{16}\right) + \log_2\left(\frac{1}{8}\right)$

B. $\log_3 27 + \ln 2$

C. $\log 100 + \log_8 8$

Refer to the function and pH scale provided in Example 6 on pg 433 to answer questions 14 to 17.

14. Calculate the pH of a solution with each hydrogen ion concentration. Round your answer to the nearest tenth.

a) 1.00 mol/L

b) 1.58×10^{-2} mol/L

c) 6.05×10^{-4} mol/L

d) 2.13×10^{-10} mol/L

e) 4.87×10^{-12} mol/L

f) 8.43×10^{-8} mol/L

15. Determine the concentration of hydrogen ions in each solution.

a) apple juice, with a pH of 3.1

b) tomato juice, with a pH of 4.2

c) milk, with a pH of 6.5

d) swimming pool water, with a pH of 7.4

e) raw egg, with a pH of 7.8

16. a) How many times more acidic is the acid used in car batteries than distilled water?

b) How many times more acidic is distilled water than an ammonia solution?

17. A sample of tap water has a pH of 6.3. How much more acidic is this tap water than distilled water, in terms of their hydrogen ion concentrations? Round your answer to the nearest tenth.

18. Sound levels are measured in decibels. The decibel scale is logarithmic and is defined by the equation

$$\beta = 10(\log I + 12)$$

where β is the sound level in decibels (dB) and I is the sound intensity in watts per square metre (W/m^2). What is the sound level, to the nearest decibel, of each sound?

a) rustle of leaves, if $I = 1 \times 10^{-11}$ W/m^2

b) siren at 30 m, if $I = 1 \times 10^{-2}$ W/m^2

c) threshold of pain, if $I = 1$ W/m^2

d) ordinary conversation at 50 cm, if $I = 3 \times 10^{-6}$ W/m^2

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19. The sound level of a loud indoor rock concert is 120 dB. The sound level of a quiet radio is 40 dB. How many times louder is the rock concert than the quiet radio? Recall that an increase of 10 dB represents an increase in loudness by a factor of 10.

20. Refer to the Richter scale in Lesson 7.2 on page 426. How much more energy was released by the earthquake that occurred near Honshu, Japan, than the earthquake that occurred in Burwash Landing, Yukon?

Closing

21. Describe how you would evaluate a logarithmic expression of the form $\log_b A$. Under what conditions could you evaluate the expression without a calculator?

Extending

22. Evaluate each logarithmic expression without technology.

a) $\log_3 \sqrt{3}$

b) $\log_4 \sqrt[3]{4}$

c) $\log_2 \sqrt{\frac{1}{8}}$

d) $\log_5 \sqrt[3]{\frac{1}{125}}$

23. German astronomer Johannes Kepler (1571–1630) determined that the time, T, it takes for a planet in our solar system to orbit the Sun is related to the average distance, r, of the planet from the Sun. If T and r are compared to the values for Earth, Kepler's result can be expressed by

$$\log T = \left(\frac{3}{2}\right) \log r - 3.264$$

where T is time in Earth years and r is the average distance from the Sun in millions of kilometres. Use this equation to determine the time it takes for each of these planets to orbit the Sun. Round your answer to the nearest hundredth of an Earth year.

a) Venus, if $r = 108$ million kilometres

b) Mars, if $r = 228$ million kilometres

c) Saturn, if $r = 1430$ million kilometres

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7 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How are the functions $y = \log x$ and $y = \ln x$ similar, and how are they different?

A: Both are logarithmic functions, but they have different bases.

The function $y = \log x$ is called the common logarithm. Its base is 10.

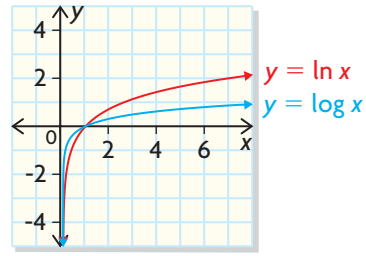
The function $y = \ln x$ is called the natural logarithm. Its base is e.

These functions share the following characteristics:

- one x-intercept at $x = 1$
- no y-intercept
- a graph that extends from quadrant IV to quadrant I
- a restricted domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
- an unrestricted range: $\{y \mid y \in \mathrm{R}\}$

Study Aid

- See Lesson 7.1.



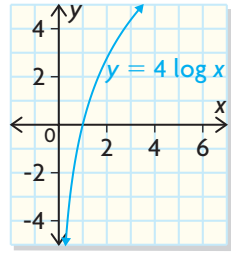
Q: How can you predict the characteristics of a logarithmic function from its equation?

A: A logarithmic function of the form

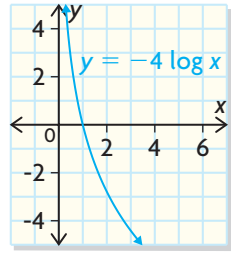
$$y = a \log x \text{ or } y = a \ln x$$

has the following characteristics:

- There is one x-intercept, and it is always $x = 1$.
- There is no y-intercept.
- The value of a indicates whether the graph of the function is increasing or decreasing and determines the end behaviour of the function:
  - The graph is increasing when $a > 0$, and it extends from quadrant IV to quadrant I.



  - The graph is decreasing when $a < 0$, and it extends from quadrant I to quadrant IV.



Study Aid

- See Lesson 7.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 2 and 3.

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Study Aid
- See Lesson 7.2, Example 1.

Q: When you evaluate a logarithmic expression, what are you determining?

A: A logarithm is an exponent. When you evaluate a logarithmic expression, you are determining the exponent that must be applied to the base of the logarithm to get the value in the expression. For example, the expression $\log_{10} 100$ defines the exponent that must be applied to a base of 10 to get the value 100.

$$\log_{10} 100 = 2$$

since

$$10^2 = 100$$

Study Aid
- See Lesson 7.2, Example 1.
- Try Mid-Chapter Review Question 6.

Q: What type of logarithmic expression can be easily evaluated with a calculator?

A: All scientific and graphing calculators can evaluate common logarithms and natural logarithms. (Some can also evaluate logarithms with different bases.) Thus, any logarithmic expression that has a base of 10 or a base of e can be evaluated with these calculators.

Study Aid
- See Lesson 7.2, Examples 3 to 5.
- Try Mid-Chapter Review Questions 7 to 9.

Q: How can you evaluate a logarithmic expression without a calculator?

A: Set the logarithmic expression equal to y. Then rewrite the expression in its equivalent exponential form. This will result in an exponential equation that you can solve. The solution to the exponential equation is the value of the logarithm. For example:

$\log_{(\frac{1}{4})} 16$

$\log_{(\frac{1}{4})} 16 = y$ Set the logarithmic expression equal to y.

$\left(\frac{1}{4}\right)^y = 16$ Rewrite the expression in exponential form, and solve for y.

$(4^{-1})^y = 4^2$ Write both sides of the equation as powers with the same base.

$4^{-y} = 4^2$ Both sides of the equation are equal only when the exponents are equal.

$-y = 2$

$y = -2$

$\log_{(\frac{1}{4})} 16 = -2$

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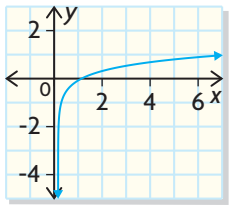
PRACTISING

Lesson 7.1

1. The graph of $y = 0.5 \log x$ is shown. State the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of this function.



2. Use each equation to predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the function. Verify your predictions with graphing technology.

a) $y = 9 \log x$ **c)** $y = 10 \ln x$

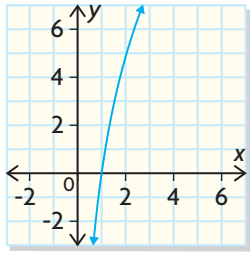
b) $y = -6 \log x$ **d)** $y = -0.3 \ln x$

3. Match each function with its corresponding graph. Provide your reasoning.

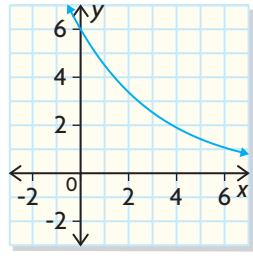
i) $y = -\log x$ **iii)** $y = 7 \log x$

ii) $y = 10^x$ **iv)** $y = 6\left(\frac{3}{4}\right)^x$

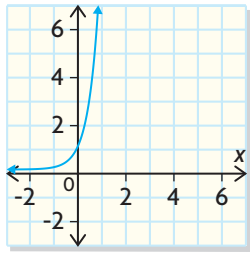
a)



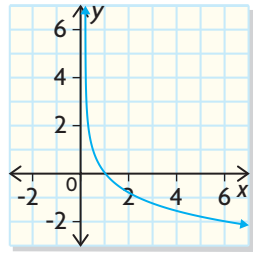
c)



b)



d)



Lesson 7.2

4. Write each logarithmic function in exponential form.

a) $y = \log_8 x$ **b)** $y = \log_3 x$

5. Write each exponential equation in logarithmic form.

a) $x = 5^y$ **b)** $x = 9^y$

6. Evaluate each logarithmic expression, to the nearest hundredth, with technology.

a) $\log 75$ **b)** $\ln 22$

7. Evaluate each logarithmic expression without technology.

a) $\log_4 16$ **c)** $\log 10$

b) $\log_3 243$ **d)** $\log_5 1$

8. Evaluate each logarithmic expression without technology.

a) $\log_3\left(\frac{1}{27}\right)$ **c)** $\log_{\left(\frac{1}{4}\right)} 64$

b) $\log_2\left(\frac{1}{4}\right)$ **d)** $\log_{\left(\frac{1}{5}\right)} 25$

9. Which expression has the greater value? Justify your decision by evaluating each expression without technology.

a) $2 \log_3 27 - \log_{\left(\frac{1}{3}\right)} 81$

b) $\log_7\left(\frac{1}{7}\right) + \log_{\left(\frac{1}{5}\right)} 125$

10. The pH of a solution, $p(x)$, can be determined using the function

$$p(x) = -\log x$$

where x represents the hydrogen ion concentration of the solution in moles/litre.

a) Determine the pH of a solution that has a hydrogen ion concentration of 0.005 mol/L.

b) Determine the concentration of hydrogen ions in distilled water, which has a pH of 7.0.

c) Black coffee has a pH of 5, and bleach has a pH of 13. In terms of their hydrogen ion concentrations, how much more acidic is black coffee than bleach?

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7.3 Laws of Logarithms

YOU WILL NEED

- graphing technology

EXPLORE...

- Which of these expressions has the greatest value?
  A. log 10 000
  B. log 10 + log 1000
  C. log 1 000 000 − log 100
  D. 2 log 100

GOAL

Develop the laws of logarithms.

INVESTIGATE the Math

You have learned that logarithms are exponents. You also know that there are laws for doing operations with exponents, called the exponent laws. Are there related laws for doing operations with logarithms?

? What are the laws of logarithms that correspond to the exponent laws?

A. Create a table like the one below.

Investigating the Product Law of Logarithms		
log 12 + log 2 = log 6 + log 4 = log 8 + log 3 =	log 24 =	log m + log n =

B. Use your calculator to evaluate each expression in the first column. Record your results. What do all three expressions have in common? Describe your observations.

C. Evaluate the expression in the second column.

i) How do the values of the expressions in the first column relate to the value of the expression in second column?

ii) How do the numbers in the expressions in the first column relate to the number in expression in the second column?

D. Based on your observations, make a conjecture for the product law of base 10 logarithms. Record your conjecture in the third column.

E. Create a table like the one below.

Investigating the Quotient Law of Logarithms		
log 12 − log 6 = log 8 − log 4 = log 6 − log 3 =	log 2 =	log m − log n =

F. Repeat parts B and C. Then make a conjecture for the quotient law of base 10 logarithms. Record your conjecture in the third column.

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G. Create a table like the one below.

Investigating the Power Law of Logarithms		
$\log 2 = \log 2^1$, $\log 2^1 =$	$1 \cdot \log 2 =$	$\log m^n =$
$\log 4 = \log 2^2$, $\log 2^2 =$	$2 \cdot \log 2 =$	
$\log 8 = \log 2^3$, $\log 2^3 =$	$3 \cdot \log 2 =$	
$\log 16 = \log 2^4$, $\log 2^4 =$	$4 \cdot \log 2 =$	

H. Repeat parts B and C. Then make a conjecture for the power law of base 10 logarithms. Record your conjecture in the third column.

I. Verify each law of logarithms that you just developed for the natural logarithm. Use technology to show that each law also applies to base e logarithms.

Reflecting

J. Are there any restrictions on the laws of logarithms that you developed? Explain.

K. Compare the operations used in each law of logarithms that you developed and its corresponding exponent law.

i) Is it possible to write any logarithm as the sum of two logarithms? Explain.

ii) Is it possible to write any logarithm as the difference of two logarithms? Explain.

iii) Is it possible to write any logarithm as the product of a real number and a logarithm? Explain.

APPLY the Math

EXAMPLE 1 | Reasoning to justify the product law for logarithms

Use exponent laws to explain why $\log 12 + \log 2$ is equivalent to $\log 24$.

Henrik's Solution

Let $x = \log 12$.
Let $y = \log 2$.

I decided to assign each logarithm in the sum to a variable.

$10^x = 12$
$10^y = 2$

I rewrote the logarithmic equations as exponential equations.

$10^x \cdot 10^y = 24$

$12 \cdot 2$ is 24, so $10^x \cdot 10^y$ must also be 24.

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$10^{(x+y)} = 24$ — Exponents are added when using the exponent law for a product.

$x + y = \log 24$ — I rewrote the exponential equation as a logarithmic equation.

$\log 12 + \log 2 = \log 24$ — I replaced the variables x and y with the logarithms that they represented.

Your Turn

Use exponent laws to explain why $\log 12 - \log 6 = \log 2$.

EXAMPLE 2 Using the laws of logarithms to evaluate logarithmic expressions

Simplify and then evaluate each logarithmic expression.

a) $\log_2 5 + \log_2 6.4$ **b)** $\log_5 100 - \log_5 4$ **c)** $\log_3 27^5$

Niklaus's Solution

a) $\log_2 5 + \log_2 6.4$ — Since the logarithms have the same base, I knew that the expression could be simplified.

$= \log_2 (5 \cdot 6.4)$ — The expression involves the sum of two logarithms, so I applied the **product law of logarithms:** $\log_b m + \log_b n = \log_b mn$

$= \log_2 32$ — The exponent that must be applied to 2 to get 32 is 5.

$= 5$

b) $\log_5 100 - \log_5 4$ — These logarithms have the same base and represent a difference, so I applied the **quotient law of logarithms:** $\log_b m - \log_b n = \log_b \left(\frac{m}{n}\right)$

$= \log_5 \left(\frac{100}{4}\right)$

$= \log_5 25$ — The exponent that must be applied to 5 to get 25 is 2.

$= 2$

c) $\log_3 27^5$ — In this expression, the base of the logarithm is 3. The power has a base of 27 and an exponent of 5.

$= 5 \log_3 27$ — Since this expression involves the logarithm of a power, I applied the **power law of logarithms:** $\log_b m^n = n \log_b m$

$= 5 \cdot 3$ — The exponent that must be applied to 3 to get 27 is 3.

$= 15$

Your Turn

Simplify and then evaluate: $\log_2 48 - \log_2 3$

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EXAMPLE 3 Writing a logarithmic expression as a single logarithm

Write each expression as a single logarithm, and then evaluate.

a) $\log_3 18 + \log_3 \left(\frac{3}{2}\right)$ **b)** $\log_5 40 - 3 \log_5 2$

Daisy's Solution

a) $\log_3 18 + \log_3 \left(\frac{3}{2}\right)$

I noticed that both logarithms have a base of 3. I decided to apply the product law of logarithms, since the expression is a sum.

$= \log_3 \left(18 \cdot \frac{3}{2}\right)$

I wrote the sum as a product.

$= \log_3 \left(\frac{18}{1} \cdot \frac{3}{2}\right)$

Then I determined the product.

$= \log_3 \left(\frac{54}{2}\right)$

$= \log_3 27$

Finally, I evaluated the logarithm. The exponent that must be applied to base 3 to get 27 is 3.

$= 3$

b) $\log_5 40 - 3 \log_5 2$

I noticed that the second logarithm is multiplied by 3. I decided to apply the power law of logarithms on the second logarithm.

$= \log_5 40 - \log_5 2^3$

I wrote the second logarithm as a power. Then I evaluated the power.

$= \log_5 40 - \log_5 8$

$= \log_5 \left(\frac{40}{8}\right)$

I noticed that both logarithms have a base of 5. I decided to apply the quotient law of logarithms, since the expression is a difference.

$= \log_5 5$

I evaluated the logarithm. The exponent that must be applied to base 5 to get 5 is 1.

$= 1$

Your Turn

Write this expression as a single logarithm, and then evaluate:

$$2 \log_3 6 + \log_3 0.75$$

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In Summary

Key Ideas

- The laws of logarithms are directly related to the exponent laws, since logarithms are exponents.
- The laws of logarithms can be used to simplify logarithmic expressions if all the terms have the same base.

Need to Know

- The laws of logarithms can be expressed as follows, where $b, m, n > 0$ and $b \neq 1$, and b, m, and n are real numbers:
  - **Product Law of Logarithms:** $\log_b mn = \log_b m + \log_b n$
    For example: $\log (5 \cdot 200) = \log 5 + \log 200$
  - **Quotient Law of Logarithms:** $\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$
    For example: $\ln \left(\frac{3}{4}\right) = \ln 3 - \ln 4$
  - **Power Law of Logarithms:** $\log_b m^n = n \log_b m$
    For example: $\log_2 5^3 = 3 \log_2 5$

CHECK Your Understanding

1. Write each of the following as a sum of logarithms.

a) $\log_5 (8 \cdot 6)$ **b)** $\log_z (xy)$

2. Write each expression as the logarithm of a product.

a) $\log_3 7 + \log_3 5$ **b)** $\log_b s + \log_b r$

3. Write each of the following as a difference of logarithms.

a) $\log_2 \left(\frac{36}{11}\right)$ **b)** $\log_z \left(\frac{x}{y}\right)$

4. Write each expression as the logarithm of a quotient.

a) $\log_3 15 - \log_3 11$ **b)** $\log_b s - \log_b r$

5. Write each of the following in the form $n \log \mathrm{b}\ m$.

a) $\log_6 7^3$ **b)** $\log_z x^y$

PRACTISING

6. Write each expression as a single logarithm, and then evaluate.

a) $\log_6 2 + \log_6 108$ **c)** $\log 5 + \log 0.002$

b) $\log_3 810 - \log_3 10$ **d)** $\log_2 16 - \log_2 2$

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7. Write each expression as a single logarithm, and then evaluate.

a) $\left(\frac{1}{2}\right)\log_3 36 - \log_3 2$

b) $\log 625 + 2\log 4$

c) $\log_2 40 - \log_2 2.5$

d) $\log_3 27 - \left(\frac{1}{3}\right)\log_3 27$

8. Write each logarithm as a sum and a difference of two logarithms.

a) $\log_2 32$

b) $\log 125$

9. Write each logarithm in question 8 as the product of an integer and a logarithm.

10. Daniel tried to use the exponent laws to explain why $\log_2 5^3 = 3\log_2 5$, but he was unable to complete his solution. Identify Daniel's error, and finish his solution.

Daniel's Solution

$y = \log_2 5$	I started by letting y represent $\log_2 5$.
$2^y = 5$	I rewrote this equation in exponential form.
$(2^y)^3 = 5^3$	I cubed both sides of the equation.
$2^{y+3} = 5^3$	I used the exponent law for a power of a power.
$y + 3 = \log_2 5^3$	Then I wrote the equation in logarithmic form. Now I'm stuck.

11. Simplify and then evaluate each logarithmic expression.

a) $\log_3 2 + \log_3 4.5$

b) $\log_3 162 - \log_3 2$

c) $\log_2 16^3$

d) $\log_5 10 - \log_5 1250$

12. Simplify and then evaluate each logarithmic expression. Round your answer to two decimal places.

a) $\log 40 - \log 8$

b) $\log 5 + 2\log 2$

13. Simplify and then evaluate: $3\log_8 2 - \left(\frac{1}{2}\right)\log_8 64$

14. Wayne made an error when he was evaluating $\log_3 27 - \log_3 9$. Identify Wayne's error, and correct his solution.

Wayne's Solution

$\log_3 27 - \log_3 9$

$= \frac{\log_3 27}{\log_3 9}$

15. Simplify and then evaluate: $\log_5 5^3 + \left(\frac{2}{3}\right)\log_5 125$

16. Simplify and then evaluate: $4\log_2 4 + \log_2 16^2$

Closing

17. Explain how each law of logarithms is related to an exponent law. Include examples in your explanations.

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Extending

18. Jarrod claims that $\log (xy)^2 = 2 \log x + 2 \log y$. Do you agree or disagree? Justify your decision.

19. Show that $\log(x^2 - 4x + 4) - \log(x^2 - 4)$ is equivalent to $\log(x - 2) - \log(x + 2)$ using two different methods.

20. Explain why this expression is valid: $\log_x x^{m-1} = m - 1$

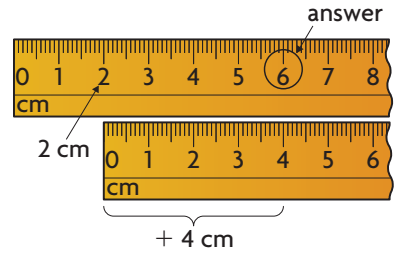
Math in Action

Making a Slide Rule

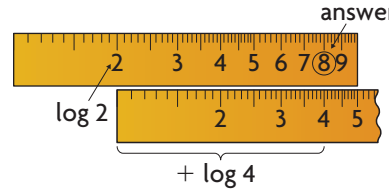
The slide rule is a mechanical calculator. It was developed in the 17th century, shortly after John Napier's invention of the logarithm. Slide rules were used in engineering, accounting, and even astronomy, since they provided a quick and effective way to multiply and divide large numbers. Some slide rules were even equipped with a magnifier to aid in reading the fine scale. These slide rules claimed precision to six decimal places.

To understand how a slide rule works, first use two standard rulers to add two numbers. For example, to determine $2 + 4$, you could align the bottom ruler at 2 on the top ruler and count 4 units on the bottom ruler. The scale on the top ruler would show the answer: 6.

Now think about using log scales to multiply two numbers, such as 2 and 4. The product of two numbers on a log scale is equivalent to the log of one of the numbers added to the log of the other number. For example, $\log (2 \cdot 4)$ is equivalent to $\log 2 + \log 4$. So, by starting at 2 on the log scale on the top ruler and then sliding the bottom ruler 4 spaces, you would get the answer 8 on the top ruler.



- Analyze the log scale above. Notice that 2 is about $\frac{1}{3}$ of the distance to 10, and 3 is about $\frac{1}{2}$ of the distance to 10.
  - **i)** What is log 2? How does it relate to the location of 2 on the log scale above?
  - **ii)** What is log 3? How does it relate to the location of 3 on the log scale above?
- Why does the log scale start at 1?
- Where would 1.5 be located on the log scale? Explain.
- Construct two log scales with measurements from 1 to 20, including decimals where possible. Slide your two log scales together, and use them to model multiplications.
- How would you use your two log scales to model a division?



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7.4 Solving Exponential Equations Using Logarithms

GOAL

Solve an exponential equation using logarithms.

YOU WILL NEED

- graphing technology

EXPLORE...

- Inflation is approximately 2.5% per year. Determine the number of years it will take for the price of milk to exceed $4.50, if it is currently $4.00.

LEARN about the Math

Yvonne has a balance of $3215 in her savings account. This account pays 2.4% interest per year, compounded annually. The compound interest formula is

$$A = P(1 + i)^n$$

where A represents the future value, P represents the principal, i represents the interest rate applied each compounding period, and n represents the number of payments.

? How long will it take for Yvonne's balance to reach $5000?

EXAMPLE 1 Solving a compound interest problem using logarithms

Dave's Solution

The principal, P, is 3215.
The interest rate, i, is 0.024.
The future value, A, is 5000.
The number of compounding periods, n, is unknown.

$$A = P(1 + i)^n$$
$$5000 = 3215(1.024)^n$$
$$\frac{5000}{3215} = (1.024)^n$$

I substituted the known values into the compound interest formula. This resulted in an exponential equation in which the variable n represents the number of compounding periods in years.

To isolate the power with the unknown exponent, n, I divided both sides of the equation by 3215.

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$$\log\left(\frac{5000}{3215}\right) = \log(1.024)^n$$

I took the common logarithm of the right side of the equation. I knew that this would allow me to use the power law of logarithms to write the resulting expression as a product of n and the log of 1.024. To maintain equality, I also took the common logarithm of the left side of the equation. By using the common logarithm, I knew that I could use my calculator to evaluate any logarithms.

$$\log\left(\frac{5000}{3215}\right) = n \log 1.024$$

I used the power law of logarithms to rewrite the expression on the right as a product of n and the log of 1.024.

$$\frac{\log\left(\frac{5000}{3215}\right)}{\log 1.024} = n$$
$$18.620... = n$$

I divided both sides of the equation by log 1.024 and solved for n.

I evaluated the left side of the equation using my calculator.

Since the interest is compounded annually, it will take 19 years for Yvonne's balance to reach \$5000.

To avoid rounding errors, I rounded only my final answer.

Reflecting

A. Verify Dave's answer using substitution.

B. Explain how the equation $64 = 2^x$ can be solved by taking the common logarithm of both sides and isolating x.

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an exponential equation

Solve the following exponential equation:

$$3^{x-1} = 20$$

Round your answer to three decimal places.

Jerry's Solution

$$3^{x-1} = 20$$

I noticed that I could not write 20 as a power of 3. This meant that I could not solve the exponential equation by writing both sides as powers with the same base.

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$$\log 3^{x-1} = \log 20$$

I knew I could take the common logarithm of both sides of the equation and still maintain the equality of the equation.

$$(x - 1)\log 3 = \log 20$$

I used the power law of logarithms to write the expression on the left as a product of $x - 1$ and the logarithm of 3.

$$\frac{(x - 1)\log 3}{\log 3} = \frac{\log 20}{\log 3}$$

I divided both sides of the equation by log 3 to isolate the expression with x.

$$x - 1 = \frac{\log 20}{\log 3}$$

$$x = \frac{\log 20}{\log 3} + 1$$

$$x = 3.726...$$

I added 1 to both sides of the equation to isolate x. Then I evaluated the right side of the equation with my calculator.

To three decimal places, $x = 3.727$.

Your Turn

Solve this exponential equation:

$$5^{x+2} = 104$$

Round your answer to three decimal places.

EXAMPLE 3 Using logarithms to solve a half-life problem

Jahmal works in a laboratory that uses radioactive substances. The laboratory received a shipment of 500 g of radioactive radon-224. Only 13.417 g of the radon-224 remained 19.0 days later. Determine the half-life of radon-224 algebraically using logarithms, and then verify your answer using graphing technology. The half-life equation is

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where A represents the amount of the substance remaining, A_0 represents the initial amount of the substance, t represents the time, and h represents the time at which only half of the substance remains.

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Karla's Solution

$A_0 = 500$
$A = 13.417$
$t = 19.0$

I recorded the given values of the variables.

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

I substituted the given values into the half-life equation.

$$13.417 = 500\left(\frac{1}{2}\right)^{\frac{19.0}{h}}$$

$$\frac{13.417}{500} = \left(\frac{1}{2}\right)^{\frac{19.0}{h}}$$

I divided both sides of the equation by 500 to isolate the expression on the right, which contains the variable exponent.

$$\log\left(\frac{13.417}{500}\right) = \log\left(\frac{1}{2}\right)^{\frac{19.0}{h}}$$

I took the common logarithm of both sides.

$$\log\left(\frac{13.417}{500}\right) = \left(\frac{19.0}{h}\right)\log\left(\frac{1}{2}\right)$$

I used the power law of logarithms to write the expression on the right as a product involving the variable and the logarithm of $\frac{1}{2}$.

$$\frac{\log\left(\frac{13.417}{500}\right)}{\log\left(\frac{1}{2}\right)} = \frac{19.0}{h}$$

I divided both sides by $\log\left(\frac{1}{2}\right)$ to isolate the expression with h.

$$5.219... = \frac{19.0}{h}$$

I evaluated the quotient of the logarithms using my calculator.

$$(5.219...)h = 19.0$$

$$h = \frac{19.0}{5.219...}$$

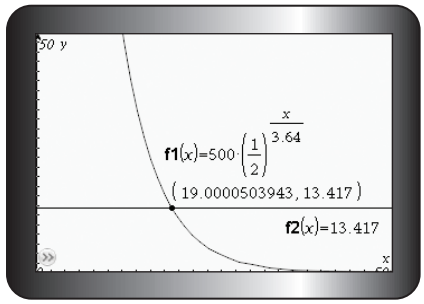
$$h = 3.639...$$

I solved for h.

The half-life of radon is about 3.6 days.

Verify:

$$f(x) = 500\left(\frac{1}{2}\right)^{\frac{x}{3.64}}$$



To verify my solution, I graphed the exponential function that represents 500 g of radon-224 decaying with a half-life of 3.64 days.

I also graphed the horizontal function $f(x) = 13.417$, which represents the amount of radon-224 remaining. I determined the point where the two functions intersect.

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According to my graph, it takes 19.0 days for 500 g of radon-224 to decay to 13.417 g.

This confirms my solution.

Your Turn

Using the half-life equation in Example 3, how long would it take for the 500 g sample of radon-224 to decay to 100 g?

EXAMPLE 4 Selecting a strategy to solve an exponential equation with different bases

Solve $2^{x-1} = 3^{x+1}$, and round your answer to three decimal places.

Lorraine's Solution

$$2^{x-1} = 3^{x+1}$$
$$\log(2^{x-1}) = \log(3^{x+1})$$

Since both sides of the equation cannot be written with the same base, I took the common logarithm of each side.

$$(x - 1)\log 2 = (x + 1)\log 3$$

I used the power law of logarithms to rewrite both sides of the equation with no exponents.

$$x\log 2 - \log 2 = x\log 3 + \log 3$$

I expanded each side using the distributive property.

$$x\log 2 - x\log 3 = \log 3 + \log 2$$

I rearranged the terms so that all the terms with x are on the left side of the equation.

$$x(\log 2 - \log 3) = \log 3 + \log 2$$

I factored x from the terms on the left side of the equation.

$$x = \frac{\log 3 + \log 2}{\log 2 - \log 3}$$
$$x = -4.419...$$

I divided both sides of the equation by $\log 2 - \log 3$ to isolate x, then evaluated the expression on the right side using my calculator.

The solution, rounded to three decimal places, is $x = -4.419$.

Your Turn

Solve $5^{x-2} - 7^{x+1} = 0$, and round your answer to three decimal places.

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EXAMPLE 5 Using a change of base strategy to evaluate a logarithm

Evaluate $\log_2 100$ to three decimal places.

Caleb's Solution

$x = \log_2 100$ — I set the expression equal to x.

$2^x = 100$ — I wrote the logarithmic equation as an exponential equation.

$\log(2^x) = \log 100$ — I took the common logarithm of each side of the equation.

$x \log 2 = \log 100$ — I used the power law of logarithms to rewrite the left side of the equation with no exponents.

$x = \dfrac{\log 100}{\log 2}$ — I isolated x.
I noticed that I started with
$$x = \log_2 100$$
but now I have
$$x = \frac{\log 100}{\log 2}$$
So the base of a logarithm can be changed from base 2 to base 10 by using the change of base formula:
$$\log_2 100 = \frac{\log 100}{\log 2}$$

$x = 6.643...$ — I evaluated the quotient with my calculator.

The value of $\log_2 100$, to three decimal places, is 6.644. — My answer is reasonable, since $2^6 = 64$ and $2^7 = 128$.

Your Turn

Evaluate $\log_5 450$ to three decimal places using the change of base formula.

Communication *Tip*

The formula
$$\log_b x = \frac{\log x}{\log b}$$
is called the change of base formula.
It can be used to determine the value of a logarithm with any base b, using common logarithms and a scientific or graphing calculator.

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In Summary

Key Idea

- If two expressions are equal, then taking the common logarithm of each expression maintains the equality of the expressions. For example, if $M = N$, then

$$\log_{10} M = \log_{10} N$$

where $M > 0$, $N > 0$, and M and N are real numbers.

Need to Know

- If an exponential equation contains one power with the unknown in the exponent, you can solve the equation algebraically by taking the common logarithm of each side.
- Sometimes an exponential equation can be solved algebraically by writing both sides of the equation with the same base, setting the exponents equal to each other, and solving for the unknown.
- Exponential equations can also be solved with graphing technology, using the system of equations strategy used for other kinds of equations.
- Some calculators will only calculate logarithms with base 10. However, you can evaluate any logarithm with base b using the change of base formula:

$$\log_b x = \frac{\log x}{\log b}$$

CHECK Your Understanding

1. Estimate the value of x in each equation, and then solve the equation. Round your answer to three decimal places.

a) $10 = 3^x$

b) $10^x = 3$

c) $27 = 4^x$

d) $78 = 5^x$

e) $4.5^x = 30$

f) $e^x = 0.4$

2. Estimate the value of x in each equation, and then solve the equation. Round your answer to three decimal places.

a) $35 = 5(3^x)$

b) $117 = 3 \times 2^x$

c) $1000 = 500(1 + 0.08)^x$

d) $45 = 100\left(\frac{1}{2}\right)^x$

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3. Estimate the value of each logarithm, and then evaluate to three decimal places using the change of base formula.

a) $\log_3 12$

b) $\log_8 3$

c) $\log_{(\frac{1}{2})} 0.1$

d) $\log_9 75$

4. Jonah has $2500 in an investment that earns 6% per year, compounded annually. Determine the number of years it will take for his balance to surpass $3500. Use the compound interest formula:

$$A = P(1 + i)^n$$

PRACTISING

5. Write each expression as a base 10 logarithm. Evaluate to three decimal places.

a) $\log_5 50$

b) $\log_2 \left(\frac{5}{8}\right)$

c) $\log_3 1000$

d) $\log_{0.1} 200$

6. Solve each equation, and round your answer to two decimal places.

a) $5^{x+1} = 24$

b) $3 = 8^{2x-1}$

c) $\left(\frac{1}{3}\right)^x - 40 = 0$

d) $\left(\frac{2}{5}\right)^{-x} = 5$

7. The intensity, I, of light passing through water can be modelled by the equation

$$I = 10^{(1-0.13x)}$$

where x represents the depth of the water in metres and I represents the intensity of the light in lux. Most aquatic plants require a light intensity of 4.2 lux for strong growth. Determine the depth of the water, to the nearest tenth of a metre, at which most aquatic plants will receive enough light.

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8. Dave thought that he could also solve the exponential equation in Example 1 by taking the logarithm of each side in the first step. However, he made an error in his solution. Correct Dave's error, and complete his solution.

Dave's Solution

$A = P(1 + i)^n$ $P = 3215$ $i = 0.024$ $A = 5000$	I substituted the given values into the compound interest formula.
The number of compounding periods, n, is unknown.	
$5000 = 3215(1.024)^n$	
$\log 5000 = \log (3215(1.024)^n)$	I took the common logarithm of each side of the equation.
$\log 5000 = n \log (3215(1.024))$	I used the power law of logarithms to rewrite the equation.
$n = \dfrac{\log 5000}{\log (3215(1.024))}$	I isolated n.
$n = 1.051...$	
It will take 2 years for the balance to reach \$5000.	The interest is compounded annually, so I rounded up to 2 years. This answer is different from my first answer, and it seems much too small.

9. Blue jeans fade when washed, due to the loss of blue dye from the fabric. Jeans with a well-worn look have, at most, 30% of the original dye in the fabric. If each washing removes about 2.2% of the original dye, how many washings are required to give a pair of jeans a well-worn look?

10. The healing of a wound with an initial area of 80 cm^2 can be modelled by the function

$$A(t) = 80(10^{-0.023t})$$

where $A(t)$ represents the area of the wound, in square centimetres, after t days of healing. In how many days will 75% of the wound be healed?

11. Damien's investment, which compounds annually, has grown from \$12 000 to \$35 000 in 18 years. Determine the doubling time, in years, for his investment.

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12. $1000 is invested at 6% per year, compounded monthly. How long, in months, will it take for the investment to triple in value? Use the compound interest formula:

$$A = P(1 + i)^n$$

13. Alex currently has $2000 in credit card debt. The interest rate on his credit card is 18.5% per year, compounded monthly. Determine the number of years it would take for Alex's debt to double if he made no payments against his balance.

14. Bisphenol A (BPA) is an organic compound that used to be found in many plastics. BPA is now considered toxic in Canada and is banned from being used to manufacture baby bottles. When Rory's blood was tested, he was found to have 50 μg (micrograms) of BPA per kilogram of body mass. Rory's blood was tested again, 3 h later. He then had 35 μg of BPA per kilogram of body mass. Determine the half-life for BPA in the blood. Round your answer to the nearest tenth of an hour.

15. Solve each equation, and round your answer to two decimal places.

a) $2^{x+1} = 3^{x-1}$

b) $6^{x-6} = 3^{x+1}$

c) $10^{x+1} = 5^{x-1}$

d) $2^{2x-1} = 5^{x+1}$

16. Solve each equation, and round your answer to two decimal places.

a) $3^{x+1} = 5^{x-1}$

b) $2^{x+3} = 3^{2x-1}$

c) $2^{x+1} - 3^{x-1} = 0$

d) $6^{x-2} = 3^{x+1}$

Closing

17. You have seen three different strategies for solving exponential equations. Describe each strategy, and illustrate how the strategy can be used to solve the equation

$$42^x = 8^{x+1}$$

Extending

18. Solve each equation, and round your answer to two decimal places.

a) $3^x - 3^{x-1} = 20$

b) $5^{x+2} + 5^x = 104$

19. Solve $2^{10^x} = e^6$, and round your answer to three decimal places.

20. Solve each equation, and round your answer to two decimal places.

a) $3^{x+2} = 5(5^{x-1})$

b) $2(2^{x+3}) = 3(3^{2x-1})$

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History | Connection

Euler's number *e*

Swiss mathematician Leonard Euler made many discoveries regarding the constant *e* and was the first person to use the letter *e* to represent it. Whether the e stands for "Euler" or is just a random letter that Euler chose is still up for debate.

Jacob Bernoulli was the first mathematician to discover a practical application involving *e*. He was investigating the effects of compound interest when the number of compounding periods was increased. He observed that, when the number of compounding periods became extremely large, the best exponential function to model this type of growth had a base of *e*.

It does not make sense to use this type of growth model when studying finance, since most investments do not have compounding periods more frequent than 365 times a year (daily). However, it does make sense as a model for natural growth. For example, if a population of rabbits grows at an average rate of 8% per year, this does not mean that eight rabbits are born at the end of every year (compounded annually) or that four rabbits are born at the end of every 6 months (compounded semi-annually). Populations of rabbits and other living things grow naturally and continuously. In other words, they have infinite compounding, and exponential functions with base *e* best model this type of growth.

- Write an equation that represents $1 growing at 100% per year, compounded annually. Calculate the balance after 1 year.
- Adjust your equation so that it represents 100% growth, compounded semi-annually. Calculate the balance after 1 year.
- Continue increasing the number of compounding periods per year and calculating year-end balances. What do you notice about the balances?
- How many times per year must compounding take place in order to have an amount equal to *e*, accurate to five decimal places?
- When modelling continuous growth, the population, *P*, can be represented by the exponential equation

  $$P = P_0 \cdot e^{kt}$$

  where P_0 is the initial population, *k* is the percent growth rate represented as a decimal, and *t* is the time in years. In 2011, the world population was estimated to be 6.91 billion and the growth rate was approximately 1.14% per year. Determine the projected global population by 2020 using this model.
- Solve the continuous growth equation for *t*. Use your new equation to predict the number of years after 2011 until the global population will have doubled.

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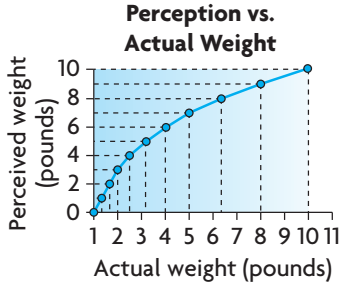
7.5 Modelling Data Using Logarithmic Functions

YOU WILL NEED

- graphing technology

EXPLORE

- In 1834, German physiologist E.H. Weber discovered that our perceptions are logarithmically related to the stimuli we receive. For example, it is more difficult for us to distinguish between the weight of two different objects if both objects are heavy than if both objects are light. To detect a change in weight, Weber found that there needs to be a larger increase in the weight of an object as the weight increases. What are the characteristics of the function that models this situation?



GOAL

Represent data using a logarithmic function and interpret the graph to solve a problem.

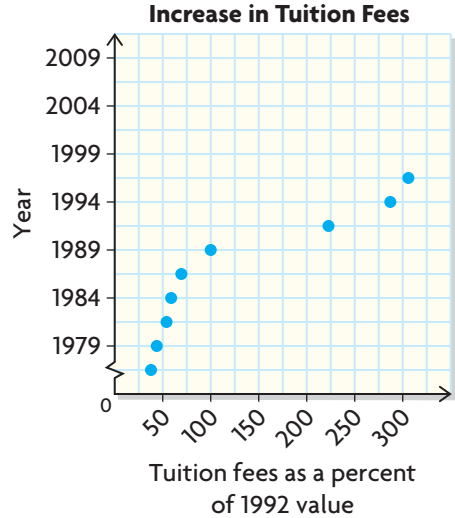
INVESTIGATE the Math

Lydia is researching the rise in tuition fees for post-secondary education in Alberta for her school website. She found some data that uses the tuition fees in 1992 as the benchmark, assigning them a value of 100%. The tuition fees in all other years are compared with the tuition fees in 1992.

Tuition Fees as a Percent of Cost in 1992 (%)	Year
37.8	1979
43.8	1982
54.0	1984
58.8	1986
69.4	1989
100.0	1992
222.7	1999
287.1	2004
305.9	2006

Statistics Canada, Table 326-0002—Consumer Price Index (CPI)

Lydia sketched a scatter plot to help her visualize the trend. She thinks that a logarithmic function may best model the data. She also thinks that writing a summary to describe the rise in tuition fees, in terms of doubling time, would be an interesting feature.



? What is the doubling time for tuition fees in Alberta?

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A. Using technology, create a scatter plot to display Lydia's data. Describe the characteristics of the trend in the data. Explain why a logarithmic function is a good model for the data.

B. Estimate the doubling time from the scatter plot. Explain how you determined your estimate.

C. Most graphing calculators and spreadsheets use the natural logarithm when performing a logarithmic regression analysis. Determine the equation of the logarithmic regression function for the data.

D. Graph your regression equation on the scatter plot. Is this regression equation a good model for the data? Explain.

E. Use your graph to estimate the doubling time for tuition fees in Alberta.

Communication *Tip*

The equation of the logarithmic regression function can be written as

$y = (\text{constant}) + (\text{multiplier}) \cdot \ln x$

Most graphing calculators and spreadsheets provide the equation of the logarithmic regression function in the form

$y = a + b \ln x$

Reflecting

F. Choose a different set of values from Lydia's data to calculate the doubling time. Is the doubling time the same as before?

G. Explain why the regression equation does not show 1992 exactly when the tuition fees were 100%.

H. Describe the intervals where the regression equation is a good model for the data and where the regression equation does not model the data very well.

APPLY the Math

EXAMPLE 1 Using logarithmic regression to solve a problem graphically

The flash on most digital cameras requires a charged capacitor in order to operate. The percent charge, Q, remaining on a capacitor was recorded at different times, t, after the flash had gone off.

The t.5 flash duration represents the time until a capacitor has only 50% of its initial charge. The t.5 flash duration also represents the length of time that the flash is effective, to ensure that the object being photographed is properly lit.

a) Construct a scatter plot for the given data.
b) Determine a logarithmic model for the data.
c) Use your logarithmic model to determine the t.5 flash duration to the nearest hundredth of a second.

Percent Charge, Q (%)	Time, t (s)
100.00	0
90.26	0.01
73.90	0.03
60.51	0.05
49.54	0.07
40.56	0.09

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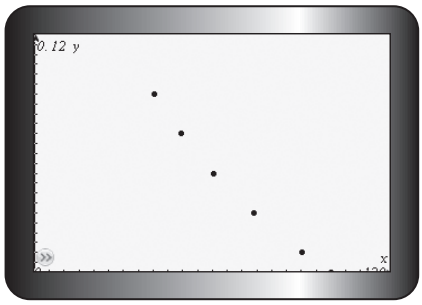
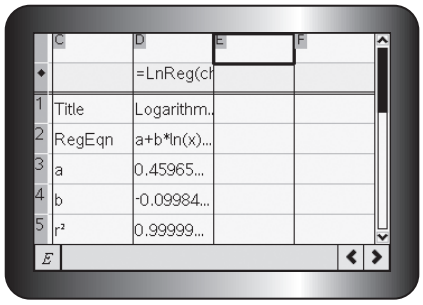
Rico's Solution

a)



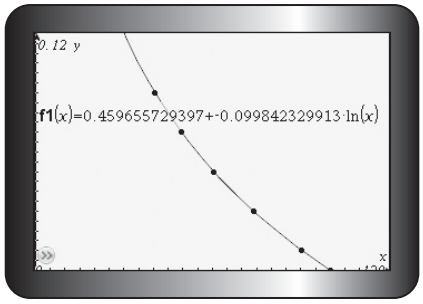
I entered the data in my graphing calculator and created a scatter plot.

b)



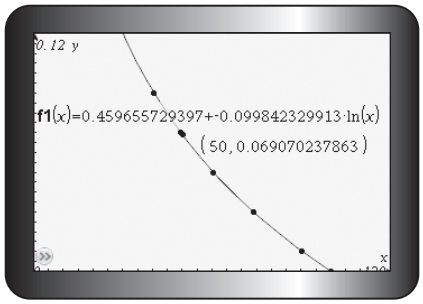
I used my calculator to determine the equation of the logarithmic regression function for the given data in order to determine the curve of best fit. I knew that my graphing calculator can only provide the equation of the logarithmic regression function using the natural logarithm in the form $y = a + b \ln x$.

The equation is $y = 0.495... - 0.099...(\ln x)$.



I verified the regression equation by entering the equation and graphing it on the same grid as the data points. I noticed that the data points lie on the graph, so the regression equation appears to be a good model for the data.

c)



To determine the t.5 flash duration for the data, I interpolated the y-value from the graph at $x = 50$.

At about 0.07 s, the t.5 flash duration has been reached.

Your Turn

The t.1 flash duration represents the time until a capacitor has just 10% of its initial charge. Determine the t.1 flash duration for the data above, to the nearest hundredth of a second.

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EXAMPLE 2 Using logarithmic regression to solve a problem algebraically

Caffeine is found in coffee, tea, and soft drinks. Many people find that caffeine makes it difficult for them to sleep. The following data was collected in a study to determine how quickly the human body metabolizes caffeine. Each person started with 200 mg of caffeine in her or his bloodstream, and the caffeine level was measured at various times.

Caffeine Level in Bloodstream, m (mg)	Time after Ingesting, t (h)	Caffeine Level in Bloodstream, m (mg)	Time after Ingesting, t (h)
168	1.0	33	14.0
167	1.5	80	7.5
113	5.0	145	3.0
145	3.0	100	6.0
90	6.5	71	8.5
125	4.0	156	2.0
138	3.5	153	2.5
77	8.0	130	4.0
83	7.0	90	6.5
50	12.0	112	5.0
150	2.5	32	16.0
55	12.0	23	18.0
112	5.0	25	17.5
84	7.0	45	13.0
136	3.5	27	18.5
180	1.0	18	20.0
110	5.0	29	15.0
75	8.0	43	12.0
76	9.0	25	17.5
49	12.5	21	19.0

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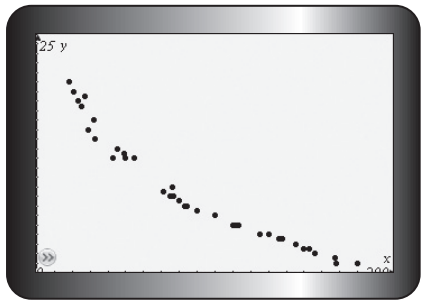
a) Determine the equation of the logarithmic regression function for the data representing time as a function of caffeine level.

b) Determine the time it takes for an average person to metabolize 50% of the caffeine in her or his bloodstream. Round your answer to the nearest tenth of an hour.

c) Paula drank a cup of coffee that contained 200 mg of caffeine at 10:00 a.m. How much caffeine will be in her bloodstream at 9:00 p.m. that evening? Round your answer to the nearest milligram.

Kourosh's Solution

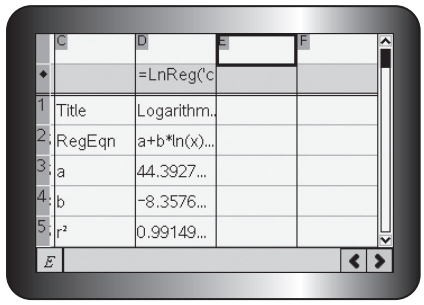
a) Let x represent the caffeine level in the bloodstream in milligrams.
Let y represent the time in hours.



I used x and y as variables since I knew that I was going to be using technology.

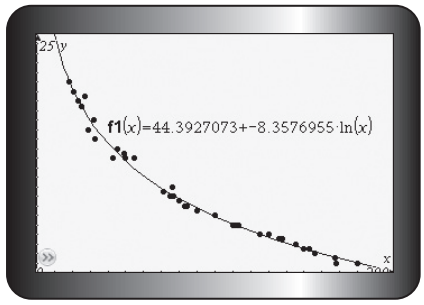
Since time is a function of caffeine level, caffeine level is the independent variable and time is the dependent variable.

I entered the data in my graphing calculator and created a scatter plot.



I used my calculator to determine the equation of the logarithmic regression function for the given data in order to determine the curve of best fit.

The equation is $y = 44.392... - 8.357... (\ln x)$.



I verified the regression equation by entering the equation and graphing it on the same grid as the data points. I noticed that, although some data points lie above or below the graph, the regression equation appears to be a good model for the data.

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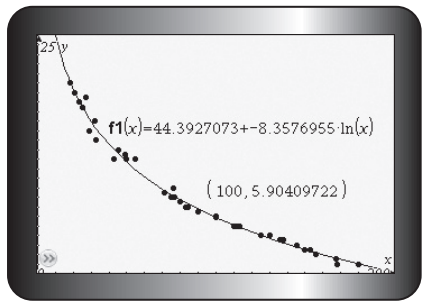
b) To determine the time it takes for an average person to metabolize 50% of the caffeine in her or his bloodstream, I need to determine 50% of the initial amount of caffeine.

$$0.5 \cdot 200 \text{ mg} = 100 \text{ mg}$$

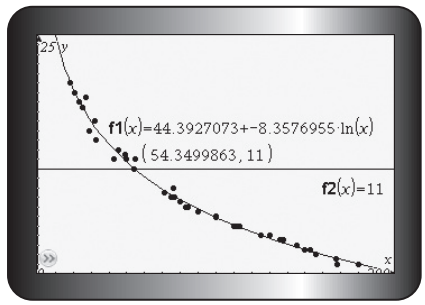


50% of the initial amount of caffeine is 100 mg, so I interpolated the y-value from the graph at $x = 100$.

It takes about 5.9 h for an average person to metabolize 50% of the caffeine in her or his bloodstream.

c) The difference between 10 a.m. and 9 p.m. is 11 h. I need to solve the equation

$$11 = 44.392... - 8.357...(\ln x).$$



I graphed the horizontal line $y = 11$ on my calculator and determined the point of intersection of the linear-logarithmic system of equations. The intersection point gives the value of the independent variable, caffeine level, when the dependent variable, time, is 11.

At 9:00 p.m., Paula will have about 54 mg of caffeine in her bloodstream.

Your Turn

Paula is only able to sleep if her caffeine levels are below 40 mg. If Paula usually goes to bed at 10 p.m., what is the latest time that she can enjoy a cup of coffee? State your answer to the nearest quarter hour.

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In Summary

Key Idea

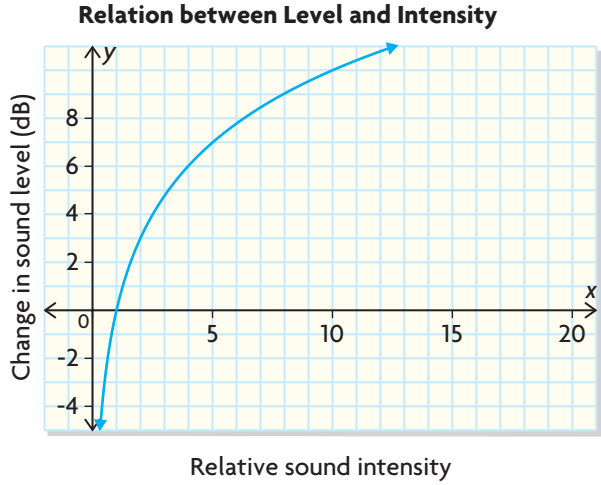
- A logarithmic function may be a good model for a set of data if the points on a scatter plot form an increasing or decreasing curve, where the domain is restricted to the set of positive real numbers.

Need to Know

- The general form of the logarithmic regression model is
$$y = (\text{constant}) + (\text{multiplier}) \cdot \ln x$$
- Most graphing calculators and spreadsheets provide the equation of the logarithmic regression function in the form
$$y = a + b \ln x$$
- A logarithmic curve of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the curve of best fit on a scatter plot or by using the equation of the logarithmic regression function.

CHECK Your Understanding

1. This logarithmic graph shows changes in the decibel level, y, as a function of the relative sound intensity, x.



a) Use the following characteristics to describe the curve, and explain why it is logarithmic:
- the location of any intercepts
- the end behaviour
- the domain and range
- whether the function is increasing or decreasing

b) By what factor does the relative sound intensity change when the decibel level increases by 10?

c) When the relative sound intensity is doubled, how does the decibel level change?

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2. Determine the equation of the logarithmic regression function that models the given data, and describe the following characteristics of the graph:
- the location of any intercepts
- the end behaviour
- the domain and range
- whether the function is increasing or decreasing

x	*y*	*x*	*y*	*x*	*y*
5	165	18	305	11	255
12	265	25	340	10	243
16	295	15	285	5	170
3	112	17	300	3	115
17	300	5	165	18	310
12	263	22	330	21	325

PRACTISING

3. Time as a function of population can be approximated by a logarithmic function. The table below shows the population of Alberta in 10-year intervals, from 1901.

Years since 1900	Population of Alberta	Years since 1900	Population of Alberta
101	2 974 807	41	796 169
91	2 545 553	31	731 605
81	2 237 724	21	588 454
71	1 627 874	11	374 295
61	1 331 944	1	73 022
51	939 501		

a) Create a scatter plot to show how time, t, is related to population, P.

b) Determine the equation of the logarithmic regression function that models the data, and describe the following characteristics of the graph:
- the location of any intercepts
- the end behaviour
- the domain and range
- whether the function is increasing or decreasing

c) Interpolate the year in which the population exceeded 2 000 000.

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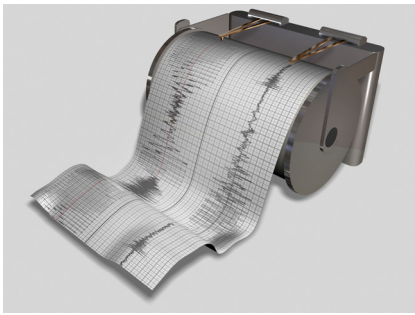

4. A seismograph records the amplitude of the vibrations during an earthquake by recording the size of the needle deflection in microns $\left(\frac{1}{1000}\text{ mm}\right)$. This data for an earthquake and its many aftershocks was recorded by a seismograph located 100 km from the earthquake.

Seismographic Reading (microns), *r*	Richter Scale Magnitude, *M*
75 023 200	7.88
2 500 010	6.40
500 320	5.70
400 250	5.60
35 400	4.55
5 005	3.70
1 053	3.02
206	2.31

a) Identify the independent and dependent variables.
b) Create a scatter plot to compare the magnitude of the earthquake to its seismographic reading.
c) Determine the equation of the logarithmic regression function that models the data in the table.
d) How many times more intense is a magnitude 5.7 earthquake than a magnitude 4.5 earthquake?

Pressure (kPa)	Altitude (m)
101.30	0
95.40	400
87.14	1000
64.50	3000
74.90	2000
39.90	6187

5. Altitude above sea level is a logarithmic function of atmospheric pressure. Michael just purchased an altimeter watch, which measures the current altitude in metres above sea level. The watch also measures the current atmospheric pressure in kilopascals (kPa). Michael recorded the atmospheric pressure at six different altitudes, as shown in the table to the left.

a) Identify the independent and dependent variables.
b) Use Michael's data to determine the equation of the logarithmic regression function for the altitude, h, as a function of the pressure, P.
c) Describe the following characteristics of the function:
- the intercepts
- the end behaviour
- the domain and range
- whether the function is increasing or decreasing

d) Michael lives at an altitude of 139 m. Determine the pressure setting that he needs to use to calibrate his watch. Round your answer to the nearest tenth.
e) Determine the atmospheric pressure at the summit of Mt. Everest, which is 8848 m above sea level, to the nearest tenth of a kilopascal.

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6. Global warming is causing glaciers to melt in the Mackenzie Mountains along the border between Yukon and the Northwest Territories. This is allowing Canadian scientists to discover hunting tools used by ancestral peoples who inhabited this region thousands of years ago. The age of some ancient objects can be estimated using carbon dating. The process involves measuring the percent of carbon-14 in the ancient object and comparing it with current values. By knowing the rate at which carbon-14 decays, scientists can then estimate the age of the object.

The age, A, in years, of an object is a logarithmic function of the percent, P, of carbon-14 remaining in the object. Data for six objects is recorded in the table.

Percent of Carbon-14, P (%)	Age, t (years)
11	18 247
18	14 176
23	12 149
28	10 523
44	6 787
63	3 819

a) Create a scatter plot to display the data, and determine the equation of the logarithmic regression function that models the data.

b) In 2000, Canadian archaeologist Tom Andrews, with members of the Shutaot'ine of the Mountain Dene Nation, discovered fragments of a birch arrow with 96.8% of its original carbon-14 still present. Estimate the age of this arrow. Round your answer to the nearest year.

c) Determine the half-life of carbon-14. Round your answer to the nearest year.

7. Jussi received an inheritance of $15 000. He invested it in a GIC that earns 4.5% per year, compounded annually. He would like his investment to accumulate to $25 000, which he can put toward a down payment on a house. The table below shows Jussi's balance over the first 5 years.

Time, t (years)	Amount, A ($)
0	15 000
1	15 675
2	16 380
3	17 117
4	17 888
5	18 693

a) Determine the equation of the exponential regression function that models this growth.

b) Determine the equation of the logarithmic regression function for time as a function of amount.

c) Use each equation to determine the time it takes for the balance to equal $25 000. Which equation do you prefer to use? Explain.

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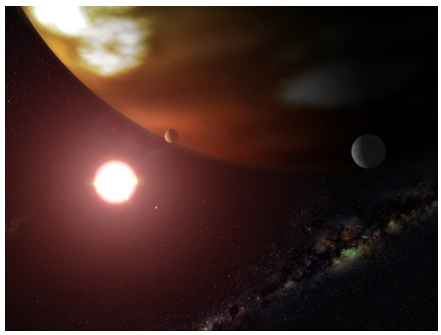
8. In astronomy, the distance to a star is determined by comparing its brightness as seen from Earth, called its apparent brightness, to its actual brightness. The distance modulus, m, is defined as the difference between the apparent brightness of a star and its standard brightness when measured from a distance of 10 pc (parsecs), where 1 pc is 3.09×10^{13} km. The distance modulus is related to the distance, d, of the star from Earth.

Star	Distance from Earth, d (pc)	Distance Modulus, m
Sun	4.85×10^{-6}	−31.57
Proxima Centauri	1.29	−4.44
Alpha Centauri	1.30	−4.40
Barnard's Star	1.82	−4.70
Sirius	2.64	−2.89
Vega	7.76	−0.55
Arcturus	11.25	0.26
Canopus	30.10	2.38
Spica	80.39	4.53
Polaris	110.00	5.19
Betelgeuse	160.00	6.01
Rigel	276.10	7.24
Deneb	490.80	8.36



a) Identify the independent and dependent variables.
b) Create a scatter plot to show the relationship between the distance modulus, m, and the distance from Earth, d.
c) Use the data to determine the equation of the logarithmic regression function for the distance modulus, m, as a function of the distance from Earth, d.
d) Wolf 359 is considered to be the fifth-closest star to Earth, at 2.39 pc. Determine its distance modulus to the nearest hundredth.
e) Gliese 876 had four known planets in 2011 and a distance modulus of −1.66. Determine the distance of this star from Earth. Round your answer to the nearest tenth.

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9. In February 2004, Mark Zuckerberg launched Facebook from his Harvard dorm room. Facebook had 1 000 000 registered users by December of that year, and it has been growing rapidly ever since.

Number of Months since Feb., 2004	Number of Registered Users (millions), *n*	Number of Months since Feb., 2004	Number of Registered Users (millions), *n*
10	1.0	62	200.0
22	5.5	65	250.0
34	12.0	67	300.0
38	20.0	70	350.0
44	50.0	72	450.0
54	100.0	77	500.0
59	150.0	79	550.0
60	175.0		

a) Create a scatter plot that relates the date, in number of months, to the number of registered users of Facebook.

b) Use the data to determine the equation of the logarithmic regression function for time, *t*, in months, as a function of the number of registered users, *n*.

c) Interpolate to determine when Facebook first surpassed 275 million registered users.

Closing

10. Describe how the dependent variable, *y*, is affected when the independent variable, *x*, is doubled or tripled in logarithmic functions of the following form:

$$y = (\text{constant}) + (\text{multiplier}) \cdot \ln x$$

Use an example in your description.

Extending

11. Lara invested $5000 at an annual interest rate of 5%. The compounding period, *n*, is related to the balance, *A*, in dollars.

a) Determine the equation of the logarithmic regression function that relates *n* to *A* for each compounding period.

i) annually **iii)** quarterly

ii) semi-annually **iv)** daily

b) Describe how the equation changes as the number of compounding periods per year increases.

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Applying Problem-Solving Strategies

Guess the Number: Binary Search

Suppose that you are searching for a word in a large alphabetical directory. One way to search is to start in the middle, read the word there, and then decide to look forward or backward within the directory based on whether the word you are looking for comes before or after the word you have read. You can repeat this process until you find the word you are looking for. Each time, the ordered list is split into two equal parts: one part is discarded, and the other part is searched further. This method is called a binary search, and it is often used by computers. It can be very efficient. For example, to locate one person in a list of the global population of around 7 billion, it would take a binary search algorithm just 33 steps. To find a random number between 1 and 1 000 000, it would take just 20 questions.

The Problem

How many questions with a yes or no answer are required to find an unknown number from 1 to 1000 using a binary search?

A. Work in pairs. Have your partner write down a secret whole number from 1 to 1000.

B. To find your partner's secret number, ask questions such as the following: "Is the number greater than 500?" Record the number of questions you asked in order to find the secret number. If you say the secret number within your question, then you have found the number. For example, if the secret number is 825 and you ask, "Is the number greater than 825?" you have found the number.

C. Switch roles several times, and repeat steps A and B.

D. On average, how many questions do you need to ask in order to find a number from 1 to 1000 using a binary search?

The Strategy

E. Write a mathematical model that describes how many numbers remain after each question in a binary search.

F. The secret number has been determined when the number of remaining numbers is 1. Solve your equation for the number of questions necessary to have one number remaining. How does this solution compare with your answer in part D?

Creating a Variation

G. How many questions are required to find a secret number from 1 to 1 000 000 000 (a billion)?

H. Is it possible to ask a different type of question that might be more efficient than a binary search? Test your method to see if it is more efficient.

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7 Chapter Self-Test

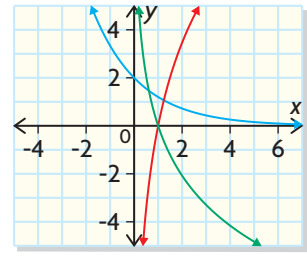
1. Use the characteristics below to describe the graph of this function:

$$y = -2 \log x$$

- the x-intercept
- the number of y-intercepts
- the end behaviour
- the domain and range
- whether the function is increasing or decreasing

2. Match each function with the corresponding graph to the right. Provide your reasoning.

i) $y = 5 \log x$ **ii)** $y = -3 \ln x$ **iii)** $y = 2(0.6)^x$



3. **a)** Write the exponential equation $x = 5^y$ in logarithmic form.
b) Write the logarithmic equation $y = \log_6 x$ in exponential form.

4. Evaluate each of the following without technology.

a) $\log_2 2^5$ **b)** $\log_{11} 121$ **c)** $\log_3 \left(\frac{1}{81}\right)$ **d)** $\log_{\left(\frac{1}{2}\right)} 8$

5. Evaluate $\log_5 \left(\frac{1}{25}\right) - \log_2 8$ without technology.

6. Simplify and then evaluate each expression.

a) $\log_2 320 - \log_2 10$ **b)** $\log_3 10 + \log_3 8.1$ **c)** $2 \log_4 8$

7. Write as a single logarithm, and then evaluate to two decimal places.

a) $3 \log 3 + 3 \log 2$

b) $4 \log 5 - \log 1$

8. Solve each equation. Round your answer to three decimal places.

a) $3^{x-1} = 215$ **b)** $5^{x+2} = 7^{x-1}$

9. Evaluate $\log_3 36$ using the change of base formula. Round your answer to two decimal places.

10. The table to the right shows the approximate energy, in kilojoules (kJ), that is released by earthquakes of different magnitudes. In 1960, the Valdivia earthquake in Chile released approximately 1.1×10^{16} kJ of energy. Use the equation of the logarithmic regression function to determine the magnitude of this earthquake to the nearest tenth.

Energy Released (kJ)	Magnitude of Earthquake
63	0
2 000	1
63 000	2
2 000 000	3
63 000 000	4
2 000 000 000	5

***WHAT** DO You Think Now?* Revisit **What Do You Think?** on page 411. How have your answers and explanations changed?

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7 Chapter Review

FREQUENTLY ASKED Questions

Study Aid
- See Lesson 7.3, Examples 1 to 3.
- Try Chapter Review Questions 3 to 8.

Q: How are logarithms related to exponents?

A: Logarithms are exponents. Recall that the exponential equation

$$x = 10^y$$

can be written as the logarithmic equation

$$y = \log x$$

In the exponential equation, y is the exponent. In the logarithmic equation, y is the logarithm of x. As a result, there are laws for operations with logarithms that are equivalent to the laws for operations with exponents.

Law	Exponents	Logarithms
Product Law	$b^m \times b^n = b^{m+n}$	$\log_b mn = \log_b m + \log_b n$
Quotient Law	$\frac{b^m}{b^n} = b^{m-n}$	$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n$
Power Law	$(b^m)^n = b^{m \times n}$	$\log_b m^n = n \log_b m$

Study Aid
- See Lesson 7.4, Examples 1 to 4.
- Try Chapter Review Questions 10 to 12.

Q: How do you know when to use logarithms to solve an equation?

A: Logarithms should be used when the unknown variable is an exponent. This situation occurs when you solve an exponential equation. It should not be confused with a situation that involves a polynomial equation in which the unknown is a base. For example:

Solve: $3^x = 10$ — This is an exponential equation.

$\log 3^x = \log 10$ — Since the variable is in the exponent, you can use logarithms to solve the equation. Take the common logarithm of each side.

$x \log 3 = \log 10$ — Then use the power law of logarithms to write the expression on the left as a product of the variable and a logarithm.

$x \log 3 = 1$ — Since $\log 10 = 1$, you can simplify the right side of the equation.

$x = \frac{1}{\log 3}$ — Isolate x and evaluate using your calculator.

$x = 2.095...$

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Solve: $x^3 = 10$ This is a cubic polynomial equation.

$\sqrt[3]{x^3} = \sqrt[3]{10}$

$x = \sqrt[3]{10}$ Since the variable is not in the exponent, you do not need logarithms to solve the equation. Instead, take the cube root of each side.

$x = 2.154...$ Solve for x.

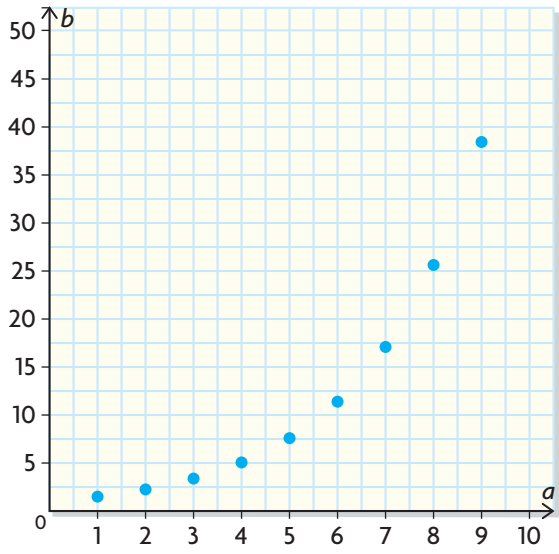
Q: To perform a logarithmic regression on a data set, how do you decide which variable is the independent variable?

A: Start by considering the context in which the data set is presented. Use the quantities in the context to help you decide which quantity depends on the other. This quantity is the dependent variable, while the other quantity is the independent variable. If you cannot decide from the context, create a scatter plot of the data. If you have chosen the independent variable correctly, you should see either an increasing or decreasing trend, with many points scattered near the y-axis. These are characteristics of a logarithmic function. If many points are scattered near the x-axis, then you have incorrectly chosen the independent variable. Switch the variables, and try again.

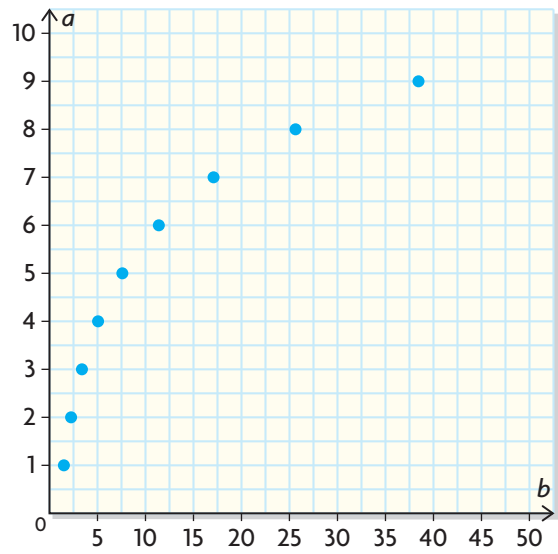
For example, the data below shows what happens when you create a scatter plot with either a or b as the independent variable. Notice that many points are scattered near the y-axis in the second scatter plot. To perform a logarithmic regression, b must be treated as the independent variable.

a	1	2	3	4	5	6	7	8	9
b	1.50	2.25	3.38	5.06	7.59	11.39	17.09	25.63	38.44

Scatter plot for a as the independent variable



Scatter plot for b as the independent variable



Study Aid

- See Lesson 7.5, Examples 1 and 2.
- Try Chapter Review Question 14.

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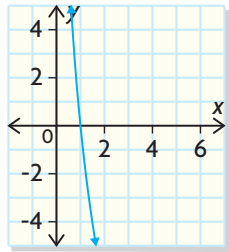
PRACTISING

Lesson 7.1

1. The graph of $y = -10 \log x$ is shown here. State the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of this function. State whether the function is increasing or decreasing.



2. Use each equation to predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the function. Verify your predictions with graphing technology.

a) $y = 25 \log x$ **c)** $y = 15 \ln x$

b) $y = -0.5 \log x$ **d)** $y = -12 \ln x$

Lesson 7.2

3. Evaluate each expression with technology. Round your answer to two decimal places.

a) $\log 200$ **b)** $\ln 5$

4. Explain what each value you determined in question 3 represents.

5. a) Write $x = 4^y$ in logarithmic form.

b) Write $y = \log_7 x$ in exponential form.

6. Evaluate each expression without technology.

a) $\log_8 64$ **c)** $\log\left(\frac{1}{100}\right)$

b) $\log_6 1$ **d)** $\log_{\left(\frac{1}{4}\right)} 16$

7. Evaluate: $2 \log_5 25 - 4 \log_5 125$

Lesson 7.3

8. Write each expression as a single logarithm, and then evaluate.

a) $\log_6 108 - \log_6 3$ **b)** $\log 4 + 2 \log 5$

9. a) State the product, quotient, and power laws of exponents. Write the corresponding law of logarithms beside each law of exponents.

b) Illustrate each law you wrote in part a) using a numerical example.

Lesson 7.4

10. Solve each exponential equation. Round your answer to three decimal places.

a) $6^x = 95$

b) $25(1.5)^x = 100$

c) $2^{2x+2} = 500$

11. Solve $3^x = 5^{x+2}$. Round your answer to three decimal places.

12. Nathan has $5500 in a savings bond that earns 4.5% per year, compounded annually. Determine the number of years it will take for his balance to surpass $7500.

13. Evaluate each expression using the change of base formula. Round your answer to two decimal places.

a) $\log_5 60$ **b)** $\log_8 146$

Lesson 7.5

14. The euphotic zone is the upper 200 m layer of the oceans. Very little sunlight penetrates deeper than 200 m, so most plants live in the euphotic zone. As a result, 70% of all photosynthesis on Earth occurs in the euphotic zone of the oceans. The following table gives light penetration data for a location in the Pacific Ocean.

Depth (m)	Penetration of Sunlight (%)
0	100.00
20	54.37
40	29.57
60	16.08
80	8.74
100	4.76

Determine the equation of the logarithmic regression function that models the data. Then use your equation to determine the amount of sunlight that penetrates to a depth of 200 m at this location. Express your answer to the nearest hundredth of a percent.

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7 Chapter Task

Leading Digits

In the early 20th century, American physicist Frank Benford observed that the leading digit of numbers in real-world data sets, such as street addresses, populations, lengths of rivers, and even heights of buildings was the digit 1 a little more than 30% of the time. This is surprising because, if the numbers were truly random, they would start with a 1 about 11% of the time, since 1 is one of nine possible digits that a number can start with. In fact, according to Benford, if the data is real, then the leading digits would occur with the probabilities shown in the table to the right.

After a more detailed analysis, Benford stated that the probability, P, of a real-world number starting with a certain digit, d, is logarithmically proportional to the expression $1 + \frac{1}{d}$.

When numbers are created at random, the leading digits do not follow Benford's law. As a result, this law has been used to identify fraudulent numbers in accounting. It has even been used as evidence when election results are suspicious.

Leading Digit, d	Probability, P (%)
1	30.1
2	17.6
3	12.5
4	9.7
5	7.9
6	6.7
7	5.8
8	5.1
9	4.6

Does real-world data truly satisfy Benford's law?

A. Decide on some real-world data sets that you would like to research. Research the data, and record at least 100 entries for each set.

B. Make a frequency table like the one below to record the frequency of each leading digit (1, 2, 3, 4, … , 9) in your data sets.

Leading Digit, d	$1 + \frac{1}{d}$	Frequency of d	Probability of d
1	$1 + \frac{1}{1} = 2$		
2	$1 + \frac{1}{2} = 1.5$		

C. Determine a logarithmic regression equation for P as a function of $1 + \frac{1}{d}$.

D. Use your regression equation to predict the probability of each digit from 1 to 9.

E. Compare your results with Benford's results.

Task Checklist

- ✔ Did you provide your data and your analysis?
- ✔ Did you provide your reasoning?
- ✔ Did you use appropriate mathematical language?

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7 Project Connection

The Final Product and Presentation

Your final presentation should be more than just a factual written report of the information you have found. To make the most of your hard work, select a format for your final presentation that will suit your strengths, as well as your topic.

Presentation Styles

To make your presentation interesting, use a format that suits your own style. Here are some ideas:

- a report on an experiment or an investigation
- a summary of a newspaper article or a case study
- a short story, musical performance, or play
- a web page
- a slide show, multimedia presentation, or video
- a debate
- an advertising campaign or pamphlet
- a demonstration or the teaching of a lesson

Here are some decisions that other students have made about the format for their presentation:

Project 1: Weather Predictions
Muhamud has researched the mathematics of weather predictions. He has decided to make his presentation a demonstration of how a weather report is prepared, including the mathematics used, followed by an actual television weather report. He plans to submit a written report on his research and conclusions, as well.

Project 2: Gender Differences
Ming has studied the differences between the responses of females and males on cognitive aptitude tests. To illustrate her findings, she will have the class complete one of the assessment tasks during her presentation and then compare the results with standardized norms. In her report, Ming plans to include testing she has done on randomly selected students at her school.

Executive Summary

Sometimes, it is effective to give your audience an executive summary of your presentation. This is a one-page summary of your presentation, which includes your research question and the conclusions you have made. Ask your teacher about making copies of your summary for the class.

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Project Example Creating a Presentation

Walther chose pests and invasive species that cause the greatest damage to natural resources in the Western provinces and territories, and the preventive measures that can be taken to control these threats as his topic. Below, he describes how he determined which format to use for his presentation.

Walther's Explanation

Most of my supporting information about the destructive nature of the predominant pests, including the mountain pine beetle, is in the form of maps, historical tables, and graphs. I decided to use a digital slide show that I can project using the interactive white board. This will help me show the data and discuss trends that I identified, including the effects of preventive measures.

I will give all of the audience members an executive summary of my research, which will include my research question, my data (with the necessary supporting visuals), and my conclusions. I will give my teacher the full report.

Evaluating Your Own Presentation

Before giving your presentation, you can use these questions to decide if it will be effective:

- Did I define my topic well? What is the best way to define my topic?
- Is my presentation focused? Will my classmates find it focused?
- Did I organize my information effectively? Is it obvious that I am following a plan in my presentation?
- Am I satisfied with my presentation? What might make it more effective?
- What unanswered questions might my audience have?

Your Turn

A. Does your topic suit some presentation formats better than others? Explain why.

B. From which presentation format do you think your audience will gain the greatest understanding? Why?

C. Choose a format and create your presentation.

D. Use the questions provided in Evaluating Your Own Presentation to assess your presentation. When you have completed your evaluation, make any changes that you think are needed.

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